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Painlevé analysis and invariant solutions of Vakhnenko–Parkes (VP) equation with power law nonlinearity

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Abstract The Vakhnenko–Parkes (VP) equation with power law nonlinearity is analyzed for Painlevé test and for Lie symmetries. The Painlevé analysis of Vakhnenko–Parkes equation with power law nonlinearity is performed by the Kruskal approach to check the Painlevé property. The Lie group formalism is also applied to derive symmetries of this equation, the ordinary differential equations deduced are further studied, and some exact solutions are obtained.

Keywords Vakhnenko–Parkes equation with power law nonlinearity · Painlevé analysis · Lie symmetries · Exact solutions

1 Introduction

One of the challenging tasks in area of applied mathematics is to look for solutions of nonlinear evolution equations (NLEEs) [1–18]. There are various types of solutions that are available in the literature for these equations. Some of them are soliton solutions, solitary wave solutions, cnoidal and snoidal waves, periodic solutions, shock wave solutions as well as various other types.

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The standard form of one of the nonlinear evolution equation (NLEE), Vakhnenko–Parkes (VP) equation, is given by

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \quad (1)$$

It can be derived from reduced Ostrovsky equation [1], which describes gravity waves propagating down a channel under the influence Coriolis force.

In this paper, we will consider the Vakhnenko–Parkes (VP) equation with power law nonlinearity [2] in following form

$$uu_{xxt} + au_x u_{xt} + bu^{2n} u_t = 0, \quad (2)$$

where a, b are nonzero real constants and n is positive integer. While taking $a = -1, b = 1$ and $n = 1$, (2) reduces to (1).

Firstly, Painlevé analysis will be performed for (2) to check its integrability. Then, Lie classical method will be used to obtain symmetries of this equation. Further using symmetries, it will be reduced to ordinary differential equations, and corresponding to the reduction, exact solutions of the Vakhnenko–Parkes (VP) equation with power law nonlinearity will be obtained.

2 Painlevé analysis of the Vakhnenko–Parkes (VP) equation with power law nonlinearity

Using the Painlevé test one can tell beforehand whether or not the given nonlinear partial differential equations are integrable. Originally, Ablowitz et al. [3] conjectured that a nonlinear partial differential equation is integrable if it has the Painlevé property: That is, its solutions are single-valued about a movable singular manifold

$$\phi(z_1, z_2, \ldots, z_n) = 0, \tag{3}$$

where ϕ is arbitrary function.

A nonlinear partial differential equation has a Painlevé property if it has a Laurent-like expansion about the movable singular manifold $\phi = 0$:

$$u(z_i) = [\phi(z_i)]^{\alpha} \sum_{j=0}^{\infty} u_j(z_i)\phi(z_i)^j, \tag{4}$$

where α is a negative integer. Here the number of arbitrary function u_j should be equal to order of the partial differential equation.

In this section, we will perform Painlevé analysis of (2).

2.1 Leading order and resonance analysis

For leading order, substitute

$$u(x,t) = u_0(x,t)\phi(x,t)^{\alpha}, \tag{5}$$

in (2) and balance the most singular or dominant terms. In this case we obtain the following value of

$$\alpha = -\frac{2}{2n-1} \tag{6}$$

and

$$u_0 = \left(-\frac{2\,(2\,n+1)\,(2\,n+a)}{b\,(2\,n-1)^2}\right)^{\frac{1}{2n-1}} \times \left(\frac{\partial}{\partial x}\phi\,(x,t)\right)^{-\frac{2}{2\,n-1}}. \tag{7}$$

Lemma *Leading-order singularity of Eq. (2) is a movable pole with $2n-1$ equal to 1 or 2. It has rational branch point for $2n-1>2$.*

The powers of ϕ at which the resonance occurs are determined by substituting

$$u(x,t) = u_0(x,t)\phi(x,t)^{\alpha} + \phi(x,t)^{\alpha+r}, \tag{8}$$

in (2) and balancing the most singular terms. Substituting (8) with (6)–(7) into (2), we obtain

$$-16\,n^2 - 4\,r^3n^2 + 12\,r^2n^2 - 4\,arn + 4\,r^3n + 4\,ar^2n - 8an - 12\,rn - 8\,n - 2\,ar^2 - 6\,ar - 2\,r - 3\,r^2 - r^3 - 4\,a = 0 \tag{9}$$

with solutions

$$r = -1,\ \frac{2\,(1+2\,n)}{-1+2\,n},\ \frac{2\,(2\,n+a)}{-1+2\,n}. \tag{10}$$

Theorem *Equation (2) does not have the Painleve property for all values of n, a with $\frac{(2\,n+a)}{-1+2\,n} \le 0$.*

Using the above lemma and theorem, let us consider the case when $a=-1$ and $n=1$, for Eq. (2), with (6)–(7), resonances occur at

$$r = -1, 2, 6. \tag{11}$$

Laurent enpension takes the form

$$u(x,t) = u_0\phi^{-2} + u_1\phi^{-1} + u_2 + u_3\phi + u_4\phi^2 + u_5\phi^3 + u_6\phi^4 \tag{12}$$

where u_6 should be arbitrary function.

Now substituting (12) into (2) with $a=-1,\ n=1$ and using the Kruskal approach [4] we have

$$\phi = x - \psi(t) \text{ and } u_j = u_j(t), \tag{13}$$

where $\psi(t)$ is arbitrary function of t.

Substituting (12) with (13) into (2) with $a = -1,\ n = 1$ and collecting the same powers of ϕ, we have

$$\begin{aligned}
u_0 &= -6\\
u_1 &= 0\\
u_2 &= u_2(t)\\
u_3 &= \frac{\dot{u_2}}{2\dot{\psi}}\\
u_4 &= \frac{-bu_2^2\dot{\psi}^3 + \ddot{u_2}\dot{\psi} - \dot{u_2}\ddot{\psi}}{10\dot{\psi}^3}\\
u_5 &= -\frac{bu_2\dot{u_2}}{6\dot{\psi}}\\
u_6 &= u_6(t),
\end{aligned} \tag{14}$$

where $\dot{}$ denotes derivative with respect to t. So Eq. (2) with $a=-1,\ n=1$ has Painlevé property.

In general, Eq. (2) does not have Painlevé property. But for truncated Larurent expansion on setting $u_2 = u_3 = u_4 = u_5 = u_6 = 0$, we have Bäcklund transformation and obtain the following solution of (2)

$$u(x,t) = \left(-\frac{2\,(2\,n+1)\,(2\,n+a)}{b\,(2\,n-1)^2}\right)^{\frac{1}{2n-1}} \times ((x-\psi(t)))^{-\frac{2}{2\,n-1}}. \tag{15}$$

3 Symmetry analysis

In this section, we will perform Lie symmetry analysis [5–15] for the Vakhnenko–Parkes (VP) Eq. (2) with power law nonlinearity.

Let us first consider the Lie group of point transformations

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2) \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned} \tag{16}$$

which leaves Eq. (2) invariant. The infinitesimal transformations (16) are such that if u is solution of Eq. (2), then u^* is also a solution. The method for determining the similarity variables of (2) mainly consists of finding the infinitesimals τ, ξ and η, which are functions of x, t, u.

The infinitesimals are determined from invariance conditions

$$\begin{aligned} &\eta u_{xxt} + u\eta^{xxt} + a(\eta^x u_{xt} + u_x\eta^{xt}) \\ &\quad + b(u^{2n}\eta^t + 2bn\eta u^{2n-1}u_t) = 0, \end{aligned} \tag{17}$$

by substituting the extended infinitesimals η^{xxt}, η^{xt}, η^x, η^t and setting the coefficients of different differentials equal to zero. We obtain a large number of PDEs in ξ, τ and η that need to be satisfied. The general solution of system of equations of PDEs in ξ, τ and η provides following forms for the infinitesimal elements

$$\begin{aligned} \xi &= C_1 + xC_2 \\ \tau &= f(t) \\ \eta &= -\frac{2}{2n-1}uC_2,\ n \neq \frac{1}{2} \end{aligned} \tag{18}$$

where C_1, C_2 are arbitrary constants and $f(t)$ is arbitrary function of t.

The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} \\ V_2 &= x\frac{\partial}{\partial x} - \frac{2}{2n-1}u\frac{\partial}{\partial u} \\ V_f &= f(t)\frac{\partial}{\partial t}. \end{aligned} \tag{19}$$

It is easy to check that $\{V_1, V_2, V_f\}$ are closed under the Lie bracket. In fact, vector fields constitute the Lie algebra as follows:

	V_1	V_2	V_f
V_1	0	V_1	0
V_2	$-V_1$	0	0
V_f	0	0	0

Remark 1 For particular case $a=-1$, $b=1$ and $n=1$, we have the symmetries obtained by Gandarias and Bruzón [15].

4 Similarity reductions and invariant solutions

To obtain the similarity reductions of Eq. (2), we have to solve the characteristic equation,

$$\frac{dx}{\xi} = \frac{dx}{\tau} = \frac{dx}{\eta}, \tag{20}$$

where ξ, τ and η are given by (18). For solving characteristic Eq. (20), we will consider two cases of vector fields: (i) $V_2 + V_f$ (ii) $V_1 + V_f$

Case (i) Vector field $V_2 + V_f$

Solving characteristic Eq. (20), we have following similarity variables

$$\begin{aligned} \sigma &= \log x - \int \frac{1}{f(t)}dt \\ u &= x^{\frac{2}{1-2n}}F(\sigma), \end{aligned} \tag{21}$$

where σ is new independent variable and F is new dependent variable.

Using (21) in Eq. (2), we have following reduction

$$\begin{aligned} &(1-2n)^2FF''' + 2(1+2n+2a)FF' \\ &\quad + (1-2n)(3+2n+2a)FF'' + 2a(1-2n)F'^2 \\ &\quad + a(1-2n)^2F'F'' + b(1-2n)^2F^{2n}F' = 0, \end{aligned} \tag{22}$$

where $'$ denotes derivative with respect to σ.

Case (ii) Vector field $V_1 + V_f$

Corresponding similarity variables are

$$\begin{aligned} \zeta &= x - \beta(t) \\ u &= G(\zeta), \end{aligned} \tag{23}$$

where $\beta(t) = \int \frac{1}{f(t)}dt$ and $\zeta, G(t)$ are new independent, dependent variables, respectively.

Using (23) in Eq. (2), we obtain

$$G'''G + aG'G'' + bG^{2n}G' = 0, \tag{24}$$

where $'$ denotes derivative with respect to ζ.

Integrating (24), we obtain

$$\frac{G^{1+2n}b}{1+2n} + \frac{a-1}{2}G'^2 + G''G = 0. \tag{25}$$

5 New exact solutions of VP equation with power law nonlinearity

Now we will consider the ODE (25) for exact solutions. If we use the transformation

$$G(\zeta) = W(\zeta)^{-\frac{2}{2n-1}} \tag{26}$$

in Eq. (25), we have

$$\frac{b}{1+2n} + \left(\frac{2a+4n}{(2n-1)^2}\right)W'^2 - \frac{2}{2n-1}WW'' = 0, \tag{27}$$

where $'$ denotes derivatives with respect to ζ.

Solving the ODE (27), we obtain the following solution

$$W(\zeta) = k_1 \pm \sqrt{-\frac{b}{2(1+2n)(a+2n)}(2n-1)\zeta}, \tag{28}$$

where k_1 is arbitrary constant.

Using (23) and (26), we obtain the following solution of main Eq. (2)

$$u(x,t) = \left(k_1 \pm \sqrt{-\frac{b}{2(1+2n)(a+2n)}(2n-1)(x-\beta(t))}\right)^{-\frac{2}{2n-1}}, \tag{29}$$

where arbitrary function $\beta(t) = \int \frac{1}{f(t)}dt$.

If we impose the condition $a = 1$, then we obtain the following solution of ODE (24)

$$G(\zeta) = \left(-\frac{2(1+2n)^2}{(\zeta - 2\zeta n - k_1 + 2k_1 n)^2 b}\right)^{\frac{1}{-1+2n}}, \tag{30}$$

where k_1 is arbitrary constant.

Corresponding solution of main Eq. (2) with condition $a = 1$ is

$$u(x,t) = \left(-\frac{2(1+2n)^2}{((x-\beta(t))(1-2n) - k_1 + 2k_1 n)^2 b}\right)^{\frac{1}{-1+2n}}, \tag{31}$$

where $\beta(t) = \int \frac{1}{f(t)}dt$.

Now let us assume the solution of (25) in the form

$$G(\zeta) = A\,\mathrm{sech}^p(B\zeta), \tag{32}$$

where A, B and p are constants to be determined.

Using (32) in (25) and balancing the nonlinear and highest derivative terms, we have

$$p = \frac{2}{2n-1}. \tag{33}$$

Now equating the coefficients of the same power terms, we obtain the following relations

$$a = -1$$
$$B = \left(\frac{b(2n-1)}{2(2n+1)}A^{2n-1}\right)^{\frac{1}{2}}. \tag{34}$$

Corresponding solutions of main Eq. (2) are given by

$$u(x,t) = A\,\mathrm{sech}^{\frac{2}{2n-1}}(B(x-\beta(t))), \tag{35}$$

with condition (34).

Remark 2 In solution (29), if we take $k_1 = 0$, we obtain the solution (15) that was obtained by using Bäcklund transformation.

Remark 3 If we consider the particular case $\beta(t) = t$ in (35), we have the same solution as obtained in [2].

6 Conclusion

In this paper, we have considered the Painlevé property and the symmetries for the Vakhnenko–Parkes (VP) equation with power law nonlinearity. Using Kruskal approach, we check the Painlevé property of VP equation with power law nonlinearity (2) and standard form of VP Eq. (1) and proved that Eq. (1) is integrable. We also obtain solution of (2) using Bäcklund transformation. Then we applied Lie classical method on (2) and using Lie symmetries reduced it into ordinary differential equations. Corresponding to reduced ordinary differential equation, we obtained some exact solutions of VP equation with power law nonlinearity (2). Solution (35) for $n = 1$, $b = 1$, $A = 1$ and $\beta(t) = AiryAi(t)$ gives Soliton solutions as shown in Fig. 1.

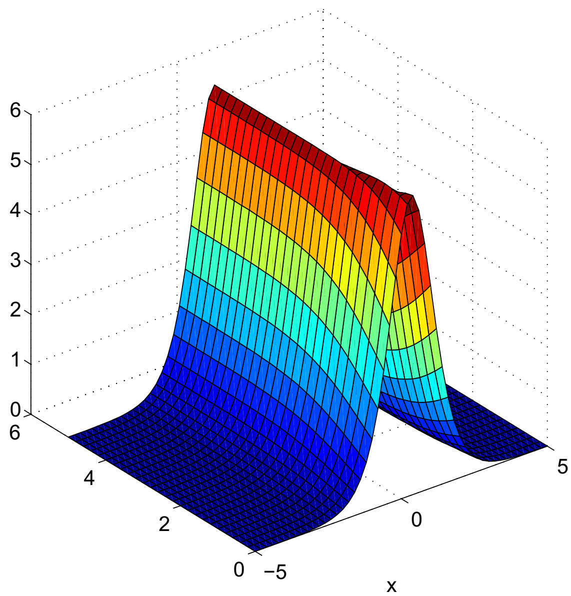

Fig. 1 Soliton solution (35) for $n = 1$, $b = 1$, $A = 1$ and $\beta(t) = AiryAi(t)$

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