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Nonclassical symmetries and similarity solutions of variable coefficient coupled KdV system using compatibility method

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Abstract The variable coefficient KdV system is investigated for nonclassical symmetries using compatibility method, and more general symmetries are reported. Several inequivalent reductions are obtained using optimal system of subalgebras, and using well-known methodologies, several traveling wave solutions are also obtained for every reduction.

Keywords Compatibility method · Symmetry analysis · Variable coefficient KdV system · Traveling wave solutions

1 Introduction

The inheriting nonlinear character of any physical phenomenon can be better understood by mean of nonlinear partial differential equations. Rich variety of powerful methods have been proposed for exact solutions, such as tanh method [1,2], *F*-expansion method [3,4], Jacobi elliptic method [5,6] and Hirota's bilinear method [7–10]. Analysis of partial differential equations using Lie group method is one of the most rigorous ways of exploiting symmetry properties of PDEs [11–16]. There exist several other methodologies for symmetry analysis of PDEs such as nonclassical method of Bluman [17–20] and direct method of Clarkson [21,22]. Apart from symmetry methods, there are several other tools available to analyze PDEs for analytical solutions [23–30], and beside analytical techniques, some numerical schemes for solving PDEs are also available [31–33]. Recently, Yan and Liu [34] proposed new method known as compatibility method for exploiting nonclassical symmetries of PDEs by establishing compatibility condition between original PDE and corresponding modified invariant surface condition. In present work, we propose to explore symmetries of following variable coefficient KdV system

$$\begin{aligned} u_t + a_1(t)\,uu_x + a_2(t)\,vv_x + a_3\,(t)u_{xxx} &= 0, \\ v_t + b_1(t)\,vu_x + b_2(t)\,uv_x + b_3(t)v_{xxx} &= 0, \end{aligned} \tag{1}$$

using compatibility method. The KdV system (1) can be derived from original KdV equation by use of transformation $u(x,t) \rightarrow u(x,t) + i\,v(x,t)$ and separating the real and imaginary parts. The KdV system (1) has been investigated by Lei et al. [35] for periodic solutions. Wronskian and multi-soliton solutions are constructed using Hirota's bilinear method [36]. Using Fréchet derivative, Singh and Gupta [37] have exploited its variable coefficient version for Lie symmetries. The coefficient version of system (1) has many forms that have been discussed in the literature, for example, Nutku–Oğuz model [38], Fuchssteiner equation [39], Drinfeld–Sokolov model [40] and Hirota–Satsuma sys-

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tem [41]. In addition to this, the constants coefficient version of system (1) has been derived as particular case of general coupled KdV system derived from the two-layer model of the atmospheric dynamical systems [42] and two-component Bose–Einstein condensates [43]. Even though in past, the KdV system (1) and its various constant coefficient version have been analyzed for Lie symmetries, we believe that using compatibility method we can recover some more general symmetries (or nonclassical symmetries in formal manner). The outline of this paper is as follows. Section 2 deals with derivation of symmetries using compatibility method. In Sect. 3, several reductions for KdV system (1) are obtained and many traveling wave solutions for these reductions using well-known methodologies are also presented, and finally, in Sect. 4 we offered conclusion.

2 Symmetries of KdV system (1)

In present work, our main aim is to obtain nonclassical symmetries of the KdV system (1) using compatibility method as described in Refs. [34,44]. Therefore, we seek nonclassical symmetry in the form

$$\begin{aligned} u_t &= r(x,t)\,u_x + s_1(x,t)\,u + p_1(x,t),\\ v_t &= r(x,t)\,v_x + s_2(x,t)\,v + p_2(x,t), \end{aligned} \tag{2}$$

where $r(x,t), s_1(x,t), s_2(x,t), p_1(x,t)$ and $p_1(x,t)$ are unknown functions that need to be determined. In order to obtain compatibility condition, we substitute (2) into (1) and results read as

$$a_1uu_x + a_2vv_x + ru_x + us_1 + a_3u_{xxx} + p_1 = 0, \tag{3a}$$

$$b_2uv_x + b_1vu_x + rv_x + vs_2 + b_3v_{xxx} + p_2 = 0, \tag{3b}$$

and thus highest order derivatives terms in (3) are u_{xxx} and v_{xxx}; below this order, no elimination could be carried. Similarly, equality of u_{tt} and v_{tt} between (1) and (2) gives

$$\begin{aligned} &a_1uu_{xt} + a_1'uu_x + va_2v_{x,t} + a_2'vv_x + a_1u_tu_x\\ &\quad + a_2v_tv_x + ru_{xt} + us_1' + a_3u_{xxxt} + r_tu_x + s_1u_t\\ &\quad + a_3'u_{xxx} + p_1' = 0, \end{aligned} \tag{4a}$$

$$\begin{aligned} &b_2uv_{xt} + b_2'uv_x + b_1vu_{xt} + b_1'vu_x + b_1u_xv_t\\ &\quad + b_2u_tv_x + rv_{xt} + vs_2' + b_3v_{xxxt} + r'v_x + s_2v_t\\ &\quad + b_3'v_{xxx} + p_2' = 0, \end{aligned} \tag{4b}$$

where $.' = \frac{d}{dt}$. Eliminating $u_t, v_t, u_{xt}, v_{xt}, u_{xxx}, v_{xxx}, u_{xxxt}$ and v_{xxxt} from (4) with the help of (2) and (3), we arrive at determining equations

$$\begin{aligned} &p_2 = 0,\quad s_{1x} = 0,\ s_{2x} = 0,\quad r_{xx} = 0,\quad p_{1t} - \frac{a_3'p_1}{a_3}\\ &\quad - 3r_xp_1 + a_3p_{1xxx} = 0,\\ &a_2' - \frac{a_3'a_2}{a_3} + 2a_2s_2 - 2a_2r_x - s_1a_2 = 0,\quad s_{1t} - \frac{a_3's_1}{a_3}\\ &\quad - 3r_xs_1 + a_1p_{1x} = 0,\quad r_t - \frac{a_3'r}{a_3} + a_1p_1\\ &\quad - 3rr_x = 0,\\ &b_1' - \frac{b_3'b_1}{b_3} + b_1s_1 - 2b_1r_x = 0,\ b_2' - \frac{b_3'b_2}{b_3} + b_2s_1\\ &\quad - 2b_2r_x = 0,\quad s_{2t} - \frac{b_3's_2}{b_3} - 3r_xs_2 + p_{1x}b_1 = 0,\\ &r_t - \frac{b_3'r}{b_3} - 3rr_x + b_2p_1 = 0,\ a_1' - \frac{a_1a_3'}{a_3} - 2a_1r_x\\ &\quad + a_1s_1 = 0, \end{aligned} \tag{5}$$

and solving the determining Eqs.(5), we get

$$\begin{aligned} &r = -\frac{a_3\left(c_1x + c_3\int a_1\,\mathrm{d}t + c_4\right)}{3c_1\int a_3\,\mathrm{d}t + c_2},\\ &p_1 = \frac{c_3a_3}{3c_1\int a_3\,\mathrm{d}t + c_2}\\ &s_1 = -\frac{c_5a_3}{3c_1\int a_3\,\mathrm{d}t + c_2},\quad s_2 = -\frac{c_6a_3}{3c_1\int a_3\,\mathrm{d}t + c_2}, \end{aligned} \tag{6}$$

where $c_i (i = 1\dots6)$ are arbitrary integration constants and a_3 remains arbitrary function of t. The coefficient functions a_1, a_2, b_1, b_2 and b_3 are governed by following constraints

$$\begin{aligned} &\left[\frac{a_1'}{a_1} - \frac{a_3'}{a_3}\right] + a_3\left[\frac{2c_1 - c_5}{3c_1\int a_3\,\mathrm{d}t + c_2}\right] = 0,\\ &\left[\frac{a_2'}{a_2} - \frac{a_3'}{a_3}\right] + a_3\left[\frac{2c_1 + c_5 - 2c_6}{3c_1\int a_3\,\mathrm{d}t + c_2}\right] = 0\\ &\left[\frac{b_1'}{b_1} - \frac{a_3'}{a_3}\right] + a_3\left[\frac{2c_1 - c_5}{3c_1\int a_3\,\mathrm{d}t + c_2}\right] = 0,\\ &\frac{b_3'}{b_3} = \frac{a_3'}{a_3},\quad b_2 = a_1. \end{aligned} \tag{7}$$

On substituting (6) into (2), the nonclassical symmetry for variable coefficient KdV system (1) is thus written as

$$u_t+\frac{a_3\left(c_1x+c_3\int a_1\,\mathrm{d}t+c_4\right)}{3c_1\int a_3\,\mathrm{d}t+c_2}u_x+\frac{c_5a_3}{3c_1\int a_3\,\mathrm{d}t+c_2}u-\frac{c_3a_3}{3c_1\int a_3\,\mathrm{d}t+c_2}\equiv 0, \quad (8a)$$

$$v_t+\frac{a_3\left(c_1x+c_3\int a_1\,\mathrm{d}t+c_4\right)}{3c_1\int a_3\,\mathrm{d}t+c_2}v_x+\frac{c_6a_3}{3c_1\int a_3\,\mathrm{d}t+c_2}v\equiv 0, \quad (8b)$$

the corresponding vector field of symmetry (8)

$$X\equiv\frac{1}{a_3}\left(3c_1\int a_3\,\mathrm{d}t+c_2\right)\frac{\partial}{\partial t}+\left(c_1x+c_3\int a_1\,\mathrm{d}t+c_4\right)\frac{\partial}{\partial x}+(-c_5u+c_3)\frac{\partial}{\partial u}-c_6v\frac{\partial}{\partial v}, \quad (9)$$

and the Lie algebra associated with vector field (9) is spanned by six vectors

$$X_1=\left(\frac{3}{a_3}\int a_3\,\mathrm{d}t\right)\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},\quad X_2=\frac{1}{a_3}\frac{\partial}{\partial t},$$
$$X_3=\left(\int a_1\,\mathrm{d}t\right)\frac{\partial}{\partial x}+u\frac{\partial}{\partial u},$$
$$X_4=\frac{\partial}{\partial x},\quad X_5=-u\frac{\partial}{\partial u},\quad X_6=-v\frac{\partial}{\partial v}. \quad (10)$$

Remark 2.1 Here we want to remark that for $c_3=0$, the nonclassical symmetries (6) reduce to symmetries obtained in [37]. This is also evident from symmetry operator (9), which is more general form of symmetry operator given in [37](see for, e.g., symmetry operator V given at (1.7) in [37]). That is, with the aid of compatibility method we able to generalize symmetries for KdV system (1).

3 Reduction of system (1) and some exact solutions

For six-dimensional Lie algebra (10), there exists infinite number of combinations of generators $X_i (i = 1\ldots6)$ for which reduction of system (1) can be achieved, but such reductions would not be inequivalent. This problem of inequivalent reductions has been duly covered in the literature [45–47] wherein for optimal system of subalgebras, Olver [45] suggested construction of adjoint table of adjoint actions between each member of Lie algebra and Ovsiannikov [46] suggested construction of global matrix of adjoint transformation. The technique of Ovsiannikov is far superior to that of Olver but is less algorithmic. Therefore, based on Olver's procedure of construction of optimal system and by solving following characteristics equation

$$\frac{\mathrm{d}x}{r}=\frac{\mathrm{d}t}{-1}=\frac{\mathrm{d}u}{-(s_1u+p_1)}=\frac{\mathrm{d}v}{-s_2v} \quad (11)$$

we have following reductions for system (1).

Reduction 3.1 $X_1+\epsilon_5X_5+\epsilon_6X_6$. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi=\frac{x^3}{\int a_3\,\mathrm{d}t},\quad u=F(\xi)\left(\int a_3\,\mathrm{d}t\right)^{-\frac{\epsilon_5}{3}},$$
$$v=G(\xi)\left(\int a_3\,\mathrm{d}t\right)^{-\frac{\epsilon_6}{3}} \quad (12)$$

and solving the constrained conditions (7), we get

$$a_1=k_1a_3\left(\int a_3\,\mathrm{d}t\right)^{-\frac{2}{3}+\frac{\epsilon_5}{3}},$$
$$a_2=k_2a_3\left(\int a_3\,\mathrm{d}t\right)^{-\frac{2}{3}-\frac{\epsilon_5}{3}+\frac{2\epsilon_6}{3}},$$
$$b_1=k_3a_3\left(\int a_3\,\mathrm{d}t\right)^{-\frac{2}{3}+\frac{\epsilon_5}{3}},\quad b_2=a_1,$$
$$b_3=k_4a_3,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$-\xi F'-\frac{1}{3}\epsilon_5F+3k_1\xi^{\frac{2}{3}}FF'+3k_2\xi^{\frac{2}{3}}GG'+27\xi^2F'''+54\xi F''+6F'=0,$$
$$-\xi G'-\frac{1}{3}\epsilon_6G+3k_3\xi^{\frac{2}{3}}GF'+3k_1\xi^{\frac{2}{3}}FG'+27k_4\xi^2G'''+54k_4\xi G''+6k_4G'=0. \quad (13)$$

The traveling wave solution for (13) can be found using simple ansätz defined as

$$F=d_0\,\xi^{\frac{1}{3}}+d_1\,\xi^{-\frac{2}{3}}$$
$$G=e_0\,\xi^{\frac{1}{3}}+e_1\,\xi^{-\frac{2}{3}} \quad (14)$$

On substituting (14) into reduced equations (13) and solving resulting algebraic equation, we obtain

$$
\begin{aligned}
d_0 &= \frac{2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4}{6k_1\,(3\,\epsilon_5 + \epsilon_6 - 2)},\\
d_1 &= -\frac{18\,(2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4)\,\epsilon_5}{k_1\,(3\,\epsilon_5+\epsilon_6-2)\left(3\,\epsilon_5{}^2+4\,\epsilon_5\epsilon_6-5\,\epsilon_5-2\,\epsilon_6+4\right)},\\
e_1 &= \frac{36\,(3\,\epsilon_5 - 4 + 2\,\epsilon_6)\,e_0}{3\,\epsilon_5{}^2 + 4\,\epsilon_5\epsilon_6 - 5\,\epsilon_5 - 2\,\epsilon_6 + 4},\\
k_2 &= \frac{(2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4)\,\epsilon_5\,(2\,\epsilon_5 - 1)}{12k_1\,(3\,\epsilon_5 + \epsilon_6 - 2)^2\,e_0{}^2}\\
k_3 &= \frac{k_1\left(4\,\epsilon_5\epsilon_6 + 2\,\epsilon_6{}^2 + \epsilon_5 - 4\,\epsilon_6\right)}{2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4},\\
k_4 &= \frac{3\,(\epsilon_6 + 1)\,\epsilon_5}{3\,\epsilon_5{}^2 + 4\,\epsilon_5\epsilon_6 - 5\,\epsilon_5 - 2\,\epsilon_6 + 4},
\end{aligned}
\tag{15}
$$

and thus using similarity transformations (12) and (14), the traveling solution for KdV system (1) can be written as,

$$
\begin{aligned}
u &= \left(d_0\left(\frac{x^3}{\int a_3(t)\,\mathrm{d}t}\right)^{\frac{1}{3}} + d_1\left(\frac{x^3}{\int a_3(t)\,\mathrm{d}t}\right)^{-\frac{2}{3}}\right)\\
&\quad\times\left(\int a_3(t)\,\mathrm{d}t\right)^{-\frac{\epsilon_5}{3}},\\
v &= \left(e_0\left(\frac{x^3}{\int a_3(t)\,\mathrm{d}t}\right)^{\frac{1}{3}} + e_1\left(\frac{x^3}{\int a_3(t)\,\mathrm{d}t}\right)^{-\frac{2}{3}}\right)\\
&\quad\times\left(\int a_3(t)\,\mathrm{d}t\right)^{-\frac{\epsilon_6}{3}},
\end{aligned}
\tag{16}
$$

where e_0 and k_1 are arbitrary; d_0, d_1, e_1, k_2, k_3 and k_4 are defined by (15).

Reduction 3.2 $X_1 + \epsilon_2 X_2$. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$
\xi = \frac{x}{(3\int a_3\,\mathrm{d}t + \epsilon_2)^{\frac{1}{3}}},\quad u = F(\xi),\quad v = G(\xi),
\tag{17}
$$

and solving the constrained conditions (7), we get

$$
\begin{aligned}
a_1 &= \frac{k_1 a_3}{(3\int a_3\,\mathrm{d}t + \epsilon_2)^{\frac{2}{3}}},\quad a_2 = \frac{k_2 a_3}{(3\int a_3\,\mathrm{d}t + \epsilon_2)^{\frac{2}{3}}},\\
b_1 &= \frac{k_3 a_3}{(3\int a_3\,\mathrm{d}t + \epsilon_2)^{\frac{2}{3}}},\quad b_2 = a_1,\quad b_3 = k_4 a_3
\end{aligned}
$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$
\begin{aligned}
&-\xi F' + k_1 F F' + k_2 G G' + F''' = 0,\\
&-\xi G' + k_3 G F' + k_1 F G' + k_4 G''' = 0.
\end{aligned}
\tag{18}
$$

For reduction (18), we seek solution in the form

$$
\begin{aligned}
F &= d_0\,\xi + d_1\\
G &= e_0\,\xi + e_0,
\end{aligned}
\tag{19}
$$

and substituting (19) into reduced equation (18) and upon solving resulting algebraic equations, we get

$$
\begin{aligned}
d_0 &= \frac{1}{k_1 + k_3},\quad e_1 = -\frac{d_1 e_0 k_1\,(k_1 + k_3)}{k_3},\\
k_2 &= \frac{k_3}{e_0{}^2\left(k_1{}^2 + 2\,k_1 k_3 + k_3{}^2\right)},
\end{aligned}
\tag{20}
$$

and substituting (20) into (19) and using similarity transformations (17), one get traveling wave solution for KdV system (1)

$$
\begin{aligned}
u &= \frac{x}{\left(3\int a_3(t)\,\mathrm{d}t + \epsilon_2\right)^{\frac{1}{3}}(k_1 + k_3)} + d_1,\\
v &= \frac{x e_0}{\left(3\int a_3(t)\,\mathrm{d}t + \epsilon_2\right)^{\frac{1}{3}}} - \frac{d_1 e_0 k_1\,(k_1 + k_3)}{k_3},
\end{aligned}
\tag{21}
$$

where d_1, e_0, k_1, k_3 and k_4 are arbitrary constants.

Reduction 3.3 $X_2 + \epsilon_6 X_6$. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$
\xi = x,\quad u = F(\xi),\quad v = G(\xi)\exp\left(-\int \epsilon_6 a_3\,\mathrm{d}t\right),
\tag{22}
$$

and solving the constrained conditions (7), we get

$$
\begin{aligned}
a_1 &= k_1 a_3,\quad a_2 = k_2 a_3 \exp\left(2\int \epsilon_6 a_3\,\mathrm{d}t\right),\\
b_1 &= k_3 a_3,\quad b_2 = k_1 a_3,\quad b_3 = k_4 a_3
\end{aligned}
$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$
\begin{aligned}
&k_1 F F' + k_2 G G' + F''' = 0\\
&k_3 G F' - \epsilon_6 G + k_1 F G' + k_4 G''' = 0.
\end{aligned}
\tag{23}
$$

For traveling wave solutions of reduction (23), we would like to use $\left(\frac{G'}{G}\right)$ method as described in [48,49].

To proceed further, we seek solution of ODEs in the form

$$F = d_m \left(\frac{G'}{G}\right)^m + d_{m-1}\left(\frac{G'}{G}\right)^{m-1} + d_{m-2}\left(\frac{G'}{G}\right)^{m-2} + \ldots,$$
$$G = e_n \left(\frac{G'}{G}\right)^n + e_{n-1}\left(\frac{G'}{G}\right)^{n-1} + e_{n-2}\left(\frac{G'}{G}\right)^{n-2} + \ldots, \tag{24}$$

where m and n can be found by homogenous balance between highest order derivative term and nonlinear term in (23); for present case, we have $m = n = 2$ and $G = G(\xi)$ satisfies second-order linear ODE

$$G'' + \lambda G' + \mu G = 0, \tag{25}$$

where λ and μ are constants. Reduction of (25) can be written as

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu} \times \left(\frac{c_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}\right)}{c_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}\right)}\right) - \frac{\lambda}{2}. \tag{26}$$

On substituting (24) into (23), an algebraic system of equations in d_0, d_1, d_2, e_0, e_1 and e_2 can be obtained by equating coefficients of $\left(\frac{G'}{G}\right)$ to zero. To save space, we have omitted this algebraic system here and solution to this system can straightforwardly be obtained using Maple as follows:

$$d_1 = \lambda d_2,\ e_0 = \left(\frac{\lambda^2}{12} + \frac{2\mu}{3}\right) e_2, \quad e_1 = \lambda e_2, \quad \epsilon_6 = 0,$$
$$k_1 = 0, \quad k_2 = -\frac{12 d_2}{{e_2}^2}, \quad k_3 = -\frac{12 k_4}{d_2} \tag{27}$$

and substituting (27) into (24) and using similarity transformations (22), we deduce that

Case 3.3.1 $\lambda^2 - 4\mu > 0$.

$$u = \left(\frac{d_2\lambda^2}{4} - d_2\mu\right) \times \left(\frac{c_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right)}{c_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right)}\right)^2 - \left(\frac{d_2\lambda^2}{4} - d_0\right)$$
$$v = \left(\frac{e_2\lambda^2}{4} - \mu e_2\right) \times \left(\frac{c_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right)}{c_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu x}\right)}\right)^2 - \left(\frac{e_2\lambda^2}{6} - \frac{2\mu e_2}{3}\right)$$

Case 3.3.2 $\lambda^2 - 4\mu < 0$.

$$u = \left(-\frac{d_2\lambda^2}{4} + d_2\mu\right) \times \left(\frac{c_1 \cos\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right) - c_2 \sin\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right)}{c_1 \sin\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right) + c_2 \cos\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right)}\right)^2 - \left(\frac{d_2\lambda^2}{4} - d_0\right)$$
$$v = \left(-\frac{e_2\lambda^2}{4} + \mu e_2\right) \times \left(\frac{c_1 \cos\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right) - c_2 \sin\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right)}{c_1 \sin\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right) + c_2 \cos\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu\xi}\right)}\right)^2 - \left(\frac{e_2\lambda^2}{6} - \frac{2\mu e_2}{3}\right)$$

Case 3.3.3 $\lambda^2 - 4\mu = 0$.

$$u = \frac{(-\lambda x c_2 - \lambda c_1 + 2 c_2)^2 d_2}{(2 x c_2 + 2 c_1)^2} + \frac{(-\lambda x c_2 - \lambda c_1 + 2 c_2) d_2 \lambda}{2 x c_2 + 2 c_1} + d_0,$$
$$v = \frac{(-\lambda x c_2 - \lambda c_1 + 2 c_2)^2 e_2}{(2 x c_2 + 2 c_1)^2} + \frac{(-\lambda x c_2 - \lambda c_1 + 2 c_2) e_2 \lambda}{2 x c_2 + 2 c_1} + \frac{(\lambda^2 + 8\mu) e_2}{12},$$

where c_1, c_2, d_0, d_2, e_2 and k_4 are arbitrary constants.

Reduction 3.4 $X_2 + \epsilon_5 X_5 + \epsilon_6 X_6$. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x, \quad u = F(\xi)\exp\left(-\int \epsilon_5 a_3\,\mathrm{d}t\right),$$
$$v = G(\xi)\exp\left(-\int \epsilon_6 a_3\,\mathrm{d}t\right), \tag{28}$$

and solving the constrained conditions (7), we get

$$a_1 = k_1 a_3 \exp\left(\int \epsilon_5 a_3\,\mathrm{d}t\right),$$
$$a_2 = k_2 a_3 \exp\left(\int (2\epsilon_6 - \epsilon_5)\,a_3\,\mathrm{d}t\right),$$
$$b_1 = k_3 a_3 \exp\left(\int \epsilon_5 a_3\,\mathrm{d}t\right),$$
$$b_2 = a_1, \quad b_3 = k_4 a_3$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$\begin{aligned} &-\epsilon_5 F + k_1 F F' + k_2 G G' + F''' = 0,\\ &-\epsilon_6 G + k_3 G F' + k_1 F G' + k_4 G''' = 0. \end{aligned} \tag{29}$$

On repeating procedure of $\left(\frac{G'}{G}\right)$ method as described in previous case of Reduction 3.3, we have following traveling wave solutions

Case 3.4.1 $\lambda^2 - 4\mu > 0$.

$$\begin{aligned} u = {}& \left(\frac{d_2\lambda^2}{4} - d_2\mu\right)\\ &\times\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right) + c_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right) + c_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right)}\right)^2\\ &-\frac{d_2\lambda^2}{6} + \frac{2d_2\mu}{3}\\ v = {}& \left(\frac{e_2\lambda^2}{4} - e_2\mu\right)\\ &\times\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right) + c_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right) + c_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu x}\right)}\right)^2\\ &-\frac{2e_2\lambda^2}{3} + \frac{2e_2\mu}{3} \end{aligned}$$

Case 3.4.2 $\lambda^2 - 4\mu < 0$.

$$\begin{aligned} u = {}& \left(-\frac{d_2\lambda^2}{4} + d_2\mu\right)\\ &\times\left(\frac{c_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu x}\right) - c_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu x}\right)}{c_1\sin\left(\frac{1}{2}\sqrt{\lambda^2+4\,\mu x}\right) + c_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu x}\right)}\right)^2\\ &-\frac{d_2\lambda^2}{6} + \frac{2d_2\mu}{3}\\ v = {}& \left(-\frac{e_2\lambda^2}{4} + e_2\mu\right)\\ &\times\left(\frac{c_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu x}\right) - c_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu x}\right)}{c_1\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu x}\right) + c_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu x}\right)}\right)^2\\ &-\frac{2e_2\lambda^2}{3} + \frac{2e_2\mu}{3} \end{aligned}$$

Case 3.4.3 $\lambda^2 - 4\mu = 0$.

$$\begin{aligned} u = {}& \frac{(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2)^2\,d_2}{(2\,\xi\,c_2 + 2\,c_1)^2}\\ &+\frac{(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2)\,d_2\lambda}{2\,\xi\,c_2 + 2\,c_1} + \frac{d_2\lambda^2}{12} + \frac{2d_2\mu}{3},\\ v = {}& \frac{(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2)^2\,e_2}{(2\,\xi\,c_2 + 2\,c_1)^2}\\ &+\frac{(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2)\,e_2\lambda}{2\,\xi\,c_2 + 2\,c_1}\\ &+\frac{e_2\lambda^2}{12} + \frac{2e_2\mu}{3}, \end{aligned}$$

where c_1, c_2, d_2 and e_2 are arbitrary constants.

Reduction 3.5 $X_2 + \epsilon_4 X_4$. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x - \int \epsilon_4 a_3\,\mathrm{d}t, \quad u = F(\xi), \quad v = G(\xi), \tag{30}$$

and solving the constrained conditions (7), we get

$$a_1 = k_1 a_3, \quad a_2 = k_2 a_3, \quad b_1 = k_3 a_3,$$
$$b_2 = a_1, \quad b_3 = k_4 a_3,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$\begin{aligned} &-\epsilon_4 F' + k_1 F F' + k_2 G G' + F''' = 0\\ &-\epsilon_4 G' + k_3 G F' + k_1 F G' + k_4 G''' = 0. \end{aligned} \tag{31}$$

Motivated by Riccati equation mapping method [50], we seek solution in the form

$$F = \sum_{i=-m}^{i=m} d_i \phi^i(\xi), \quad G = \sum_{i=-n}^{i=n} e_i \phi^i(\xi), \tag{32}$$

where $\phi'(\xi) = r + p\phi(\xi) + q\phi^2(\xi)$. In (31) by balancing the highest order derivative term and nonlinear term, we have $m = n = 2$, and on substituting (32) into (31), a set of algebraic equations are obtained by equating powers of $\phi(\xi)$ to zero. Solving such system yields

$$\begin{aligned} &d_{-2} = d_{-1} = 0,\\ &d_0 = -\frac{p^2k_1k_4 + 8\,qrk_1k_4 - \epsilon_4 k_1 - \epsilon_4 k_3}{(k_1+k_3)\,k_1},\\ &d_1 = -\frac{12\,pqk_4}{k_1+k_3},\ d_2 = -\frac{12q^2k_4}{k_1+k_3},\ e_{-2} = e_{-1} = 0,\\ &e_0 = k\left(\frac{p^2+8\,qr}{k_1+k_3}\right),\ e_1 = \left(\frac{12pqk}{k_1+k_3}\right),\\ &e_2 = \frac{12kq^2}{k_1+k_3},\ k = \left(\frac{-k_1{k_4}^2 + k_1k_4 + k_3k_4}{k_2}\right)^{\frac{1}{2}} \end{aligned} \tag{33}$$

Using solution set (33) and similarity transformations (30), we have following traveling wave solutions

$$u = d_0 + d_1\phi_i(\xi) + d_2\phi_i^2(\xi),\quad v = e_0 + e_1\phi_i(\xi) + e_2\phi_i^2(\xi), \tag{34}$$

where for $i = 1, 2, 3, 4$.

$$\phi_1(\xi) = -\frac{p}{2q} - \frac{\sqrt{\theta}\tanh\left(\frac{1}{2}\sqrt{\theta}\xi\right)}{2q}, \qquad \theta > 0, \tag{35a}$$

$$\phi_2(\xi) = \frac{4r\sin\left(\frac{1}{4}\sqrt{-\theta}\xi\right)\cos\left(\frac{1}{4}\sqrt{-\theta}\xi\right)}{-2\,p\sin\left(\frac{1}{4}\sqrt{-\theta}\xi\right)\cos\left(\frac{1}{4}\sqrt{-\theta}\xi\right) + 2\sqrt{-\theta}\left(\cos\left(\frac{1}{4}\,\theta\xi\right)\right)^2 - \sqrt{-\theta}},\quad \theta < 0, \tag{35b}$$

$$\phi_3(\xi) = -\frac{p\,(\cosh(p\xi) + \sinh(p\xi))}{q\,(d_1 + \cosh(p\xi) + \sinh(p\xi))}, \quad r = 0, \tag{35c}$$

$$\phi_4(\xi) = -\frac{1}{q\xi + d_2}, \quad r = p = 0, \tag{35d}$$

where $\theta = p^2 - 4\,qr$ and $\xi = x - \int \epsilon_4 a_3\,\mathrm{d}t$, k_1, k_2, k_3, k_4 and ϵ_4 all are arbitrary constants.

Reduction 3.6 $X_2 + \epsilon_4 X_4 + \epsilon_5 X_5 + \epsilon_6 X_6$. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x - \int \epsilon_4 a_3\,\mathrm{d}t,\quad u = F(\xi)\exp\left(-\int \epsilon_5 a_3\,\mathrm{d}t\right),\quad v = G(\xi)\exp\left(-\int \epsilon_6 a_3\,\mathrm{d}t\right), \tag{36}$$

and solving the constrained conditions (7), we get

$$\begin{aligned} a_1 &= k_1 a_3 \exp\left(\int \epsilon_5 a_3\,\mathrm{d}t\right),\\ a_2 &= k_2 a_3 \exp\left(\int (2\epsilon_6 - \epsilon_5) a_3\,\mathrm{d}t\right)\\ b_1 &= k_3 a_3 \exp\left(\int \epsilon_5 a_3\,\mathrm{d}t\right),\\ b_2 &= k_1 a_3 \exp\left(\int \epsilon_5 a_3\,\mathrm{d}t\right),\quad b_3 = k_4 a_3, \end{aligned}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$\begin{aligned} &-\epsilon_4 F' - \epsilon_5 F + k_1 FF' + k_2 GG' + F''' = 0\\ &-\epsilon_4 G' - \epsilon_6 G + k_3 GF' + k_1 FG' + k_4 G''' = 0. \end{aligned} \tag{37}$$

For reduction of type (37), we seek solution in the form

$$\begin{aligned} F &= d_0 + d_1\tanh(\xi) + d_2\tanh^2(\xi)\\ G &= e_0 + e_1\tanh(\xi) + e_2\tanh^2(\xi), \end{aligned} \tag{38}$$

and substituting (38) into (37) and solving resulting algebraic equation, we get

$$d_1 = 0,\ d_2 = -\frac{12k_4}{k_1+k_3},\ e_0 = -\frac{8k}{k_1+k_3},\ e_1 = 0,$$
$$e_2 = \frac{12k}{k_1+k_3},\ k = \left(\frac{-k_1{k_4}^2 + k_1k_4 + k_3k_4}{k_2}\right)^{\frac{1}{2}}$$
$$\epsilon_4 = \frac{k_1\left(d_0k_1 + d_0k_3 - 8\,k_4\right)}{k_1+k_3},\ \epsilon_5 = 0,\ \epsilon_6 = 0. \quad (39)$$

Thus, with the aid of solutions (39) and similarity transformations (36), traveling wave solution for KdV system (1) can be written as

$$u = d_0 - \frac{12k_4}{k_1+k_3} \times \left(\tanh\left(x - \int \frac{k_1\left(d_0k_1+d_0k_3-8\,k_4\right)a_3\left(t\right)}{k_1+k_3}\,\mathrm{d}t\right)\right)^2$$
$$v = -\frac{8k}{k_1+k_3} + \frac{12k}{k_1+k_3} \times \left(\tanh\left(x - \int \frac{k_1\left(d_0k_1+d_0k_3-8\,k_4\right)a_3\left(t\right)}{k_1+k_3}\,\mathrm{d}t\right)\right)^2, \quad (40)$$

where d_0, k_1, k_3 and k_4 are arbitrary constants and k is given in (39).

Reduction 3.7 X_2. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x, \quad u = F(\xi), \quad v = G(\xi), \quad (41)$$

and solving the constrained conditions (7), we get

$$a_1 = k_1a_3, \quad a_2 = k_2a_3, \quad b_1 = k_3a_3,$$
$$b_2 = a_1, \quad b_3 = k_4a_3,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$\begin{aligned} &k_1FF' + k_2GG' + F''' = 0 \\ &k_3GF' + k_1FG' + k_4G''' = 0. \end{aligned} \quad (42)$$

For traveling wave solutions of reduction (42), we use same procedure of Riccati equation mapping method as described in Reduction 3.5, and after balancing the highest order derivative term with that of non-linear term, we seek solution of (42) in the form

$$\begin{aligned} &F = d_0 + d_1\phi(\xi) + d_2\phi^2(\xi), \\ &G = e_0 + e_1\phi(\xi) + e_2\phi^2(\xi), \end{aligned} \quad (43)$$

where $\phi'(\xi) = r + p\phi(\xi) + q\phi^2(\xi)$. Substitution of (43) into (42) yields

$$d_1 = \frac{d_2p}{q},$$
$$e_0 = -\frac{e_2\left(p^2d_2 - 12\,q^2d_0 + 8\,qrd_2 - d_0d_2k_3\right)}{k_3{d_2}^2},$$
$$e_1 = \frac{e_2p}{q},\ k_1 = -\frac{12\,q^2 + d_2k_3}{d_2},$$
$$k_2 = \frac{k_3{d_2}^2}{{e_2}^2},\ k_4 = 1, \quad (44)$$

and using (44) and similarity transformations (41), traveling wave solutions for KdV system (1) can be written as

$$\begin{aligned} &u = d_0 + d_1\phi_i(\xi) + d_2\phi_i^2(\xi), \\ &v = e_0 + e_1\phi_i(\xi) + e_2\phi_i^2(\xi), \end{aligned} \quad (45)$$

where for $i = 1, 2, 3, 4$, $\phi_i(\xi)$ are already defined in (35) and d_0, d_2, e_2 and k_3 are arbitrary.

Reduction 3.8 X_4. The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = t, \quad u = F(\xi), \quad v = G(\xi), \quad (46)$$

and solving the constrained conditions (7), we get

$$a_1 = \frac{k_1a_3}{\left(\int a_3\,\mathrm{d}t\right)^{\frac{2}{3}}}, \quad a_2 = \frac{k_2a_3}{\left(\int a_3\,\mathrm{d}t\right)^{\frac{2}{3}}},$$
$$b_1 = \frac{k_3a_3}{\left(\int a_3\,\mathrm{d}t\right)^{\frac{2}{3}}}, \quad b_2 = a_1, \quad b_3 = k_4a_3$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration, and the reduction of system (1) is obtained as follows:

$$F' = 0, \quad G' = 0 \quad (47)$$

and reduction (47) is equivalent to constant solution of system (1).

Remark 3.1 All the calculations are handled using Maple program, and the traveling wave solutions that we have reported are verified using same tool.

4 Conclusion

The variable coefficient coupled KdV system has been investigated in this manuscript using recently introduced compatibility method. We have successfully applied procedure of compatibility method with the aid of Maple program without which bulky algebraic equations could have been unmanageable. Bypassing the invariance criterion of Lie group method, six-dimensional symmetry algebra is constructed. From symmetry operator (9) for KdV system (1), it is evident that in this work, we able to recover more general symmetries than reported in earlier work [37]. Pairwise disjointness of subalgebras through adjoint action has given optimal system for symmetry algebra, and consequently, several inequivalent reductions and their traveling wave solutions are reported.

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