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Symmetry analysis, explicit power series solutions and conservation laws of the space-time fractional variant Boussinesq system

Baljinder Kour and Sachin Kumar[a]
Department of Mathematics and Statistics, Central University of Punjab, Bathinda - 151001, Punjab, India

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Abstract. In this study, the classical Lie symmetry method is successfully applied to investigate the symmetries of the space-time fractional variant Boussinesq system which was introduced as a model of water waves. With the help of the obtained symmetries, the governing system is reduced into the system of nonlinear fractional ordinary differential equations (NLFODEs) which contains Erdèlyi-Kober fractional differential operators via Riemann-Liouville fractional derivative. The system is also studied for the explicit power series solution. The obtained power series solution is further examined for the convergence. The conservation laws of the governing system are constructed by using the new conservation theorem and generalization of the Noether operators. The numerical approximation for the fractional system is also found by using the residual power series method (RSPM). Some figures are also presented to explain the physical understanding for both explicit and approximate solutions.

1 Introduction

The branch of mathematics that deals with derivatives and integrals of arbitrary order real or complex numbers is called "fractional calculus". The research area of nonlinear fractional order partial differential equations (NLFPDEs) has been remarkably active for the past few decades. The physical phenomena in various engineering and scientific fields, specifically statistical mechanics, fluid flow, viscoelasticity, material science, biosciences, medicine, geology, electrochemistry, electromagnetics described by differential models is examined for the fractional order. There are many methods to find solutions of NLFPDEs such as the Homotopy method [1], G'/G the expansion method [2], the sub-equation method [3], the exp function method [4] and the Lie symmetry method [5, 6]. Exact solutions of fractional differential equations (FDEs) have attracted substantial interest in the literature. A lot of information of physical phenomena modeled by differential equations has been supplied by Lie symmetry analysis and conservation laws [7]. The Lie symmetry method is systematically used in different nonlinear scientific fields and plays an important role for finding exact solutions of nonlinear partial differential equations (NLPDEs). The theory of the Lie group for obtaining the group invariant solutions of NLPDEs is one of the most powerful methods. We found lot of books and survey articles about applications of the Lie group [5, 8–12]. The investigation process of conservation laws has been given in some articles [13–15]. In order to find conservation laws the fractional order Noether operators have been generalized by using a new conservation theorem [16].

Nowadays researchers have shown much interest in space-time NLFPDEs in place of NLPDEs of integer order for a better understanding of nonlinear phenomena. There have been many approaches for the solution of fractional partial differential equations (FPDEs) [14, 17–19]. In this paper, we study the system of a space-time fractional variant Boussinesq system for Lie symmetry reduction, explicit power series solution and conservation laws.

The Boussinesq equation was originally introduced to describe the continuation of long waves in the shallow water [20–22]. In this work, we consider the space-time fractional variant Boussinesq system,

$$\begin{aligned}
&\frac{\partial^\gamma H}{\partial t^\gamma} + H\frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^\beta U}{\partial x^\beta} = 0,\\
&\frac{\partial^\gamma U}{\partial t^\gamma} + H\frac{\partial^\beta U}{\partial x^\beta} + U\frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^3 H}{\partial x^3} = 0,
\end{aligned} \tag{1}$$

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[a] e-mail: `sachin1jan@yahoo.com (corresponding author)`

is a mathematical model which describes the propagation of long surface waves, where H is the velocity and U is the height of the free wave surface for fluid in the trough. The Boussinesq equations have been studied by the various authors by many different approaches. In [22], the authors studied the Boussinesq equation for soliton solutions. In [23], the traveling wave solution of the variant Boussinesq is found. In [20, 24] the variant Boussinesq system has been studied for Lie symmetry reductions, exact solutions and conservation laws.

The outline of the paper is as follows: In sect. 2, the Lie symmetry method is introduced to deal with a system of space-time FPDEs. The Lie symmetry method is employed to obtain symmetry reduction of the considered system of space-time fractional equations in sect. 3. Section 4 contains explicit power series solutions of governing system (1). In sect. 5, the explicit power series is analysed for convergence. Conservation laws are found for the governing system in sect. 6. The residual power series method (RPSM) is applied to find a residual approximate solution in sect. 7. Section 8 contains the numerical result and discussion of the approximate solution by interesting graphs of the approximate solution. Finally, in sect. 9, concluding remarks are presented.

2 Symmetry analysis of space-time fractional partial differential equations

In this section, we present a brief study of the Lie symmetry analysis for FPDEs with two independent variables (x,t) and two dependent functions (v_1,v_2). Consider a scalar space-time FPDEs of the following form:

$$\begin{aligned} &F_1\left(x,t,\frac{\partial^\gamma v_1}{\partial t^\gamma},\frac{\partial^\beta v_1}{\partial x^\beta},\frac{\partial^\beta v_2}{\partial x^\beta},\frac{\partial v_1}{\partial x},\frac{\partial v_2}{\partial x},\frac{\partial^2 v_1}{\partial x^2},\dots\right)=0,\\ &F_2\left(x,t,\frac{\partial^\gamma v_2}{\partial t^\gamma},\frac{\partial^\beta v_2}{\partial x^\beta},\frac{\partial^\beta v_1}{\partial x^\beta},\frac{\partial v_2}{\partial x},\frac{\partial v_1}{\partial x},\frac{\partial^2 v_2}{\partial x^2},\dots\right)=0,\end{aligned} \tag{2}$$

where for $i=1,2$. Here, $\frac{\partial v_i}{\partial x}$ and $\frac{\partial^2 v_i}{\partial x^2}$ denote the derivatives of order 1 and 2, respectively, and $\frac{\partial^\gamma v_i}{\partial t^\gamma}$, $\frac{\partial^\beta v_i}{\partial x^\beta}$ are the Riemann-Liouville fractional derivatives of orders γ, $\beta>0$, respectively, defined by

$$\frac{\partial^\gamma f(x,t)}{\partial t^\gamma}=\begin{cases}\dfrac{1}{\Gamma(1+[\gamma]-\gamma)}\dfrac{\partial^{1+[\gamma]}}{\partial t^{1+[\gamma]}}\displaystyle\int_0^t (t-s)^{[\gamma]-\gamma}f(x,s)\mathrm{d}s, & t>0,\ [\gamma]<\gamma<[\gamma]+1,\\ \dfrac{\partial^n f(x,t)}{\partial t^n}, & \gamma=[\gamma]=n\in\mathbb{N},\end{cases} \tag{3}$$

where $\Gamma(\gamma)$ is Euler's gamma function. Let us suppose that the above FPDEs, eqs. (2), are invariant under a one parameter (ϵ) Lie group of transformations

$$\begin{aligned} &x^*=x+\varepsilon\xi(x,t,v_1,v_2)+O(\varepsilon^2), && t^*=t+\varepsilon\tau(x,t,v_1,v_2)+O(\varepsilon^2),\\ &v_1^*=v_1+\varepsilon\eta(x,t,v_1,v_2)+O(\varepsilon^2), && v_2^*=v_2+\varepsilon\phi(x,t,v_1,v_2)+O(\varepsilon^2),\\ &\frac{\partial^\gamma v_1^*}{\partial t^\gamma}=\frac{\partial^\gamma v_1}{\partial t^\gamma}+\varepsilon\eta^{\gamma,t}+O(\varepsilon^2), && \frac{\partial^\gamma v_2^*}{\partial t^\gamma}=\frac{\partial^\gamma v_2}{\partial t^\gamma}+\varepsilon\phi^{\gamma,t}+O(\varepsilon^2),\\ &\frac{\partial^\beta v_1^*}{\partial x^\beta}=\frac{\partial^\beta v_1}{\partial x^\beta}+\varepsilon\eta^{\beta,t}+O(\varepsilon^2), && \frac{\partial^\beta v_2^*}{\partial x^\beta}=\frac{\partial^\beta v_2}{\partial x^\beta}+\varepsilon\phi^{\beta,t}+O(\varepsilon^2),\\ &\frac{\partial v_1^*}{\partial x}=\frac{\partial v_1}{\partial x}+\varepsilon\eta^{x}+O(\varepsilon^2), && \frac{\partial v_2^*}{\partial x}=\frac{\partial v_2}{\partial x}+\varepsilon\phi^{x}+O(\varepsilon^2),\\ &\frac{\partial^2 v_1^*}{\partial x^2}=\frac{\partial^2 v_1}{\partial x^2}+\varepsilon\eta^{xx}+O(\varepsilon^2), && \frac{\partial^2 v_2^*}{\partial x^2}=\frac{\partial^2 v_2}{\partial x^2}+\varepsilon\phi^{xx}+O(\varepsilon^2),\\ &\quad\vdots, \end{aligned} \tag{4}$$

where ξ, τ, η, ϕ are the infinitesimals; $\eta^{\gamma,t}$ and $\phi^{\gamma,t}$ are the extended infinitesimals of order γ; $\eta^{\beta,x}$ and $\phi^{\beta,x}$ are the extended infinitesimals of order β; η^x, ϕ^x, η^{xx} and ϕ^{xx} are the integer order extended infinitesimals, which leave the system of eqs. (2) invariant. The infinitesimal transformation (4) are such that if v_1, v_2 is the solution of (2) then so is v_1^*, v_2^*. The associated infinitesimal generator is given by

$$X=\xi(x,t,v_1,v_2)\frac{\partial}{\partial x}+\tau(x,t,v_1,v_2)\frac{\partial}{\partial t}+\eta(x,t,v_1,v_2)\frac{\partial}{\partial v_1}+\phi(x,t,v_1,v_2)\frac{\partial}{\partial v_2}\,. \tag{5}$$

The prolonged infinitesimal generator is as

$$pr^{(\gamma,\beta)}X = X + \eta^{\gamma,t}\partial_{\partial_t^{\gamma} v_1} + \phi^{\gamma,t}\partial_{\partial_t^{\gamma} v_2} + \eta^{\beta,x}\partial_{\partial_x^{\beta} v_1} + \phi^{\beta,x}\partial_{\partial_x^{\beta} v_2} + \eta^{x}\partial_{v_{1,x}} + \phi^{x}\partial_{v_{2,x}} + \eta^{xx}\partial_{v_{1,xx}} + \eta^{xxx}\partial_{v_{1,xxx}} + \dots, \quad (6)$$

where $\partial_{x_1}^{\gamma} v_i = \frac{\partial^{\gamma} v_i}{\partial x_1^{\gamma}}$, $\partial_{v_i,x_1} = \frac{\partial v_i}{\partial x_1}$, $\partial_{v_i,x_1x_1} = \frac{\partial^2 v_i}{\partial x_1^2}$ and so on, for $i = 1, 2$ and $x_1 = x, t$.

Since the lower limit of the Riemann-Liouville fractional derivative (3) with respect to x, t is fixed, therefore the invariance condition yields

$$\xi(x,t,v_1,v_2)|_{x=0} = 0, \qquad \tau(x,t,v_1,v_2)|_{t=0} = 0. \quad (7)$$

The infinitesimals are given as

$$\begin{aligned} \xi(x,t,v_1,v_2) &= \left.\frac{\mathrm{d}x^*}{\mathrm{d}\epsilon}\right|_{\epsilon=0}, \quad \tau(x,t,v_1,v_2) = \left.\frac{\mathrm{d}t^*}{\mathrm{d}\epsilon}\right|_{\epsilon=0}, \\ \eta(x,t,v_1,v_2) &= \left.\frac{\mathrm{d}v_1^*}{\mathrm{d}\epsilon}\right|_{\epsilon=0}, \quad \phi(x,t,v_1,v_2) = \left.\frac{\mathrm{d}v_2^*}{\mathrm{d}\epsilon}\right|_{\epsilon=0}, \end{aligned} \quad (8)$$

also,

$$\begin{aligned} \eta^{x} &= D_x\eta - v_{1,x}D_x\xi - v_{1,t}D_x\tau - v_{2,x}D_x\xi - v_{2,t}D_x\tau, \\ \eta^{xx} &= D_x\eta^{x} - v_{1,xx}D_x\xi - v_{1,xt}D_x\tau - v_{2,xx}D_x\xi - v_{2,xt}D_x\tau. \end{aligned} \quad (9)$$

In a similar way we can obtain η^{xxx}, ϕ^{x}, ϕ^{xx} and so on, where D_x is the total derivative given by

$$D_x = \frac{\mathrm{d}}{\mathrm{d}x} + v_{1,x}\frac{\mathrm{d}v_1}{\mathrm{d}x} + v_{2,x}\frac{\mathrm{d}v_2}{\mathrm{d}x} + v_{1,xx}\frac{\mathrm{d}v_{1,x}}{\mathrm{d}x} + v_{2,xx}\frac{\mathrm{d}v_{2,x}}{\mathrm{d}x} + \dots. \quad (10)$$

The γ-th–order extended infinitesimal $\eta^{\gamma,t}$ related to the Riemann-Liouville fractional derivative is

$$\eta^{\gamma,t} = D_t^{\gamma}(\eta) + \xi D_t^{\gamma}(v_{1,x}) - D_t^{\gamma}(\xi v_{1,x}) + \tau D_t^{\gamma}(v_{1,t}) - D_t^{\gamma}(\tau v_{1,t}). \quad (11)$$

Now by applying the generalized Leibnitz rule [25, 26], eq. (11) can be expanded as

$$\eta^{\gamma,t} = D_t^{\gamma}(\eta) - \gamma D_t\tau\frac{\partial^{\gamma} v_1}{\partial t_{\gamma}} - \sum_{n=1}^{\infty}\binom{\gamma}{n}D_t^{n}(\xi)D_t^{\gamma-n}(v_1,x) - \sum_{n=1}^{\infty}\binom{\gamma}{n+1}D_t^{n+1}(\tau)D_t^{\gamma-n}(v_1), \quad (12)$$

where D_t represents the total derivative operator. Now by using the generalized chain rule and the generalized Leibnitz rule $D_t^{\gamma}(\eta)$ can be obtained as

$$\begin{aligned} D_t^{\gamma}(\eta) = {}& \frac{\partial^{\gamma}\eta}{\partial t^{\gamma}} + \left(\eta_{v_1}\frac{\partial^{\gamma} v_1}{\partial t^{\gamma}} - v_1\frac{\partial^{\gamma}\eta_{v_1}}{\partial t^{\gamma}}\right) + \left(\eta_{v_2}\frac{\partial^{\gamma} v_2}{\partial t^{\gamma}} - v_2\frac{\partial^{\gamma}\eta_{v_2}}{\partial t^{\gamma}}\right) + \mu_{\eta,\gamma,1} + \mu_{\eta,\gamma,2} + \sum_{n=1}^{\infty}\binom{\gamma}{n}\frac{\partial^{n}\eta_{v_2}}{\partial t^{n}}D_t^{\gamma-n}(v_2) \\ & + \sum_{n=1}^{\infty}\binom{\gamma}{n}\frac{\partial^{n}\eta_{v_1}}{\partial t^{n}}D_t^{\gamma-n}(v_1), \end{aligned} \quad (13)$$

where

$$\mu_{\eta,\gamma,i} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=2}^{k-1}\binom{\gamma}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\gamma}}{\Gamma(n-\gamma+1)}(-v_i)^{r}\frac{\partial^{m}}{\partial t^{m}}(v_i^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial v_i^{k}}, \quad i = 1, 2 \quad (14)$$

and $\eta_{v_i} = \frac{\partial\eta}{\partial v_i}$.

After substituting the value of $D_t^{\gamma}(\eta)$ in (12), the γ-th–order extended infinitesimal $\eta^{\gamma,t}$ for the system of FPDEs of the form (2) is

$$\begin{aligned} \eta^{\gamma,t} = {}& \frac{\partial^{\gamma}\eta}{\partial t^{\gamma}} + (\eta_{v_1} - \gamma D_t(\tau))\frac{\partial^{\gamma} v_1}{\partial t^{\gamma}} - v_1\frac{\partial^{\gamma}\eta_{v_1}}{\partial t^{\gamma}} + \left(\eta_{v_2}\frac{\partial^{\gamma} v_2}{\partial t^{\gamma}} - v_2\frac{\partial^{\gamma}\eta_{v_2}}{\partial t^{\gamma}}\right) - \sum_{n=1}^{\infty}\binom{\gamma}{n}D_t^{n}(\xi)D_t^{\gamma-n}(v_1,x) \\ & + \sum_{n=1}^{\infty}\binom{\gamma}{n}\frac{\partial^{n}\eta_{v_2}}{\partial t^{n}}D_t^{\gamma-n}(v_2) + \mu_{\eta,\gamma,1} + \mu_{\eta,\gamma,2} + \sum_{n=1}^{\infty}\left[\binom{\gamma}{n}\frac{\partial^{n}\eta_{v_1}}{\partial t^{n}} - \binom{\gamma}{n+1}D_t^{n+1}(\tau)\right]D_t^{\gamma-n}(v_1). \end{aligned} \quad (15)$$

Similar to this expression of $\eta^{\gamma,t}$, the γ-th extended infinitesimal $\phi^{\gamma,t}$, β-th extended infinitesimal $\eta^{\beta,x}$ and β-th extended infinitesimal $\phi^{\beta,x}$ are obtained easily.

3 Symmetries and reduction of space-time fractional variant Boussinesq system

In this section, the Lie symmetries of the system (1), will be obtained and further using that symmetries the system (1) will be reduced into system of NLFODEs.

The invariance criteria of system (1) under the one parameter Lie group of transformations is described by the following theorems:

Theorem 1. *The group of transformations* (4)*, introduces the Lie symmetries of the system* (1) *as follows:*

$$\xi = c_1 x, \qquad \tau = \frac{(3+\beta)}{2\gamma} c_1 t, \qquad \eta = \frac{(\beta-3)}{2} c_1 H, \qquad \phi = (\beta-3) c_1 U, \tag{16}$$

where c_1 *is an arbitrary constant and the Lie symmetry generator of the system* (1) *is obtained as*

$$X = x\frac{\partial}{\partial x} + \frac{(3+\beta)}{2\gamma} t \frac{\partial}{\partial t} + \frac{(\beta-3)H}{2}\frac{\partial}{\partial H} + (\beta-3)U\frac{\partial}{\partial U}\,. \tag{17}$$

Proof. The invariance criteria of system (1) under the one parameter Lie group of transformations are obtained as

$$\begin{aligned}
&\eta^{\gamma,t} + H\eta^{\beta,x} + \eta\partial_x^\beta H + \phi^{\beta,x} = 0,\\
&\phi^{\gamma,t} + U\eta^{\beta,x} + \phi\partial_x^\beta H + H\phi^{\beta,x} + \eta\partial_x^\beta U + \eta^{xxx} = 0,
\end{aligned} \tag{18}$$

where η^{xxx} is the extended infinitesimal of order 3, $\eta^{\gamma,t}$, $\phi^{\gamma,t}$ are extended infinitesimals of order γ, and $\eta^{\beta,x}$, $\phi^{\beta,x}$ are extended infinitesimals of order β.

Now substituting the values of the prolongations and equating the coefficient of various powers of partial derivatives of H and U to zero, for $0 < \gamma$, $\beta \le 1$ the determining equations are obtained as

$$\begin{aligned}
&\xi_t = \xi_H = \xi_U = 0,\\
&\tau_x = \tau_H = \tau_U = 0,\\
&\eta_U = 0,\\
&\eta_H - 3\xi_x - \phi_U + \gamma D_t\tau = 0,
\end{aligned}$$

$$\begin{aligned}
&\eta + H(\gamma D_t\tau - \beta D_x\xi) = 0,\\
&\eta_H - \phi_U - \gamma D_t\gamma + \beta D_x\xi = 0,\\
&\partial_t^\gamma\eta - H\partial_t^\gamma\eta_H = 0,\\
&\partial_t^\gamma\phi - U\partial_t^\gamma\phi_U + \eta_{xxx} = 0,\\
&\binom{\gamma}{n}\partial_t^n\eta_H - \binom{\gamma}{n+1}D_t^{n+1}\tau = 0, \quad n\in\mathbb{N},\\
&\binom{\gamma}{n}\partial_t^n\phi_U - \binom{\gamma}{n+1}D_t^{n+1}\tau = 0, \quad n\in\mathbb{N},\\
&\binom{\beta}{n}\partial_x^n\eta_H - \binom{\beta}{n+1}D_x^{n+1}\xi = 0, \quad n\in\mathbb{N},\\
&\binom{\beta}{n}\partial_x^n\phi_U - \binom{\beta}{n+1}D_x^{n+1}\xi = 0, \quad n\in\mathbb{N}.
\end{aligned} \tag{19}$$

Solving all PDEs and FPDEs together in the system (19), also using (7), we obtained the infinitesimals

$$\xi = c_1 x, \qquad \tau = \frac{(3+\beta)}{2\gamma} c_1 t, \qquad \eta = \frac{(\beta-3)}{2} c_1 H, \qquad \phi = (\beta-3) c_1 U, \tag{20}$$

where c_1 is the arbitrary constant.

From these infinitesimals (20), the symmetry generator (17) of system (1 is obtained. Thus, the auxiliary equations are

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{\frac{(3+\beta)}{2\gamma}t} = \frac{\mathrm{d}H}{\frac{(\beta-3)}{2}H} = \frac{\mathrm{d}U}{(\beta-3)U}\,. \tag{21}$$

The symmetry reduction of the system (1) is obtained in the following theorem.

Theorem 2. *The similarity transformations of the system with similarity variable* $y = xt^{-\frac{2\gamma}{\beta+3}}$ *are* $H(x,t) = t^{\frac{\gamma(\beta-3)}{\beta+3}} f(y)$ *and* $U(x,t) = t^{\frac{2\gamma(\beta-3)}{\beta+3}} h(y)$ *and the corresponding symmetry reduction of the system of NLFPDEs* (1) *for* γ, $\beta > 0$ *into the system of NLFODEs is given by*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{\beta+3}{\gamma}}^{1-\frac{6\gamma}{\beta+3},\gamma} f\right)(y) + y^{-\beta}\left(\left(\mathcal{Q}_1^{-\beta,\beta} h\right)(y) + f(y)\left(\mathcal{Q}_1^{-\beta,\beta} f\right)(y)\right) = 0,\\
&\left(\mathcal{P}_{\frac{\beta+3}{\gamma}}^{1+\frac{\gamma(\beta-9)}{\beta+3},\gamma} h\right)(y) + y^{-\beta}\left(h(y)\left(\mathcal{Q}_1^{-\beta,\beta} f\right)(y) + f(y)\left(\mathcal{Q}_1^{-\beta,\beta} h\right)(y)\right) + f'''(y) = 0,
\end{aligned} \tag{22}$$

where $(\mathcal{P}_\varrho^{\vartheta,\gamma})$ *is the left-hand–sided Erdèlyi-Kober fractional differential operator is given by*

$$\begin{aligned}
&\left(\mathcal{P}_\varrho^{\vartheta,\gamma} f\right)(y) = \prod_{k=0}^{r-1}\left(\vartheta + k - \frac{1}{\varrho} y \frac{\mathrm{d}}{\mathrm{d}y}\right)\left(\mathcal{M}_\varrho^{\vartheta+\gamma,r-\gamma} f\right)(y), \quad y > 0,\ \varrho > 0,\ \gamma > 0,\\
&r = \begin{cases} [\gamma] + 1 & \text{if } \gamma \notin \mathbb{N},\\ \gamma & \text{if } \gamma \in \mathbb{N}, \end{cases}
\end{aligned} \tag{23}$$

where

$$\left(\mathcal{M}_\varrho^{\vartheta,\gamma} f\right)(y) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_1^\infty (\rho - 1)^{\gamma-1} \rho^{-(\vartheta+\alpha)} f\left(y\rho^{\frac{1}{\varrho}}\right) \mathrm{d}\rho & \text{if } \gamma > 0,\\ f(y) & \text{if } \gamma = 0 \end{cases}$$

is the left-hand–sided Erdèlyi-Kober fractional integral operator.

Also, $(\mathcal{Q}_\varrho^{\vartheta,\beta})$ *is the right-hand–sided Erdèlyi-Kober fractional differential operator given as*

$$\begin{aligned}
&\left(\mathcal{Q}_\varrho^{\vartheta,\beta} f\right)(y) = \prod_{k=1}^{r}\left(\vartheta + k + \frac{1}{\varrho} y \frac{\mathrm{d}}{\mathrm{d}y}\right)\left(\mathcal{T}_\varrho^{\vartheta+\beta,r-\beta} f\right)(y), \quad y > 0,\ \varrho > 0,\ \beta > 0,\\
&r = \begin{cases} [\beta] + 1 & \text{if } \beta \notin \mathbb{N},\\ \beta & \text{if } \beta \in \mathbb{N}, \end{cases}
\end{aligned} \tag{24}$$

where

$$\left(\mathcal{T}_\varrho^{\vartheta,\beta} f\right)(y) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^1 (1 - \rho)^{\beta-1} \rho^{\vartheta} f\left(y\rho^{\frac{1}{\varrho}}\right) \mathrm{d}\rho & \text{if } \beta > 0,\\ f(y) & \text{if } \beta = 0 \end{cases}$$

is the right-hand–sided Erdèlyi-Kober fractional integral operator.

Proof. First, we find the value of the fractional partial derivatives $\frac{\partial^\beta H}{\partial x^\beta}$ and $\frac{\partial^\beta U}{\partial x^\beta}$ of order $\beta > 0$.

For $n - 1 < \beta < n$; $n \in \mathbb{N}$, then by the definition of the Riemann-Liouville fractional derivative (3), we have

$$\frac{\partial^\beta H}{\partial x^\beta} = \frac{\partial^n}{\partial x^n} \frac{1}{\Gamma(n-\beta)} \int_0^x (x - s)^{n-\beta-1} t^{\frac{\gamma(\beta-3)}{\beta+3}} f\left(t^{-\frac{2\gamma}{\beta+3}} s\right) \mathrm{d}s. \tag{25}$$

Let $\omega = \frac{s}{x}$, and using similarity variable $y = xt^{-\frac{2\gamma}{\beta+3}}$, we get following equation:

$$\begin{aligned}
\frac{\partial^\beta H}{\partial x^\beta} &= \frac{\partial^n}{\partial x^n} \frac{t^{\frac{\gamma(\beta-3)}{\beta+3}}}{\Gamma(n-\beta)} \int_0^1 (1-\omega)^{n-\beta-1} x^{n-\beta} f(y\omega) \mathrm{d}\omega\\
&= t^{\frac{\gamma(\beta-3)}{\beta+3}} \frac{\partial^n}{\partial x^n} x^{n-\beta} \left(\mathcal{T}_1^{0,n-\beta} f\right)(y),
\end{aligned} \tag{26}$$

where

$$\left(\mathcal{T}_1^{0,n-\beta} f\right)(y) = \frac{1}{\Gamma(n-\beta)} \int_0^1 (1-\omega)^{n-\beta-1} f(y\omega) \mathrm{d}\omega \tag{27}$$

is the right-hand–sided Erdèlyi-Kober fractional integral operator (25).

For $\beta = n = 1, 2, 3, \ldots$, the relation (26) holds for $(\mathcal{T}_1^{\vartheta,0} f)(y) = f(y)$.

Now for greater simplification, let $\rho(y)$ be a continuously differential function for $y = xt^{-\frac{2\gamma}{\beta+3}}$, then we have the following relation:

$$x\frac{\partial}{\partial x}\rho(y) = x\frac{\mathrm{d}}{\mathrm{d}y}\rho(y)t^{-\frac{2\gamma}{\beta+3}} = y\frac{\mathrm{d}}{\mathrm{d}y}\rho(y). \tag{28}$$

Thus, it follows:

$$\begin{aligned}\frac{\partial^n}{\partial x^n}\left(x^{n-\beta}\left(\mathcal{T}_1^{0,n-\beta}f\right)(y)\right) &= \frac{\partial^{n-1}}{\partial x^{n-1}}\left[x^{n-\beta-1}\left(n-\beta+y\frac{\mathrm{d}}{\mathrm{d}y}\right)\left(\mathcal{T}_1^{0,n-\beta}f\right)(y)\right],\\ &= \ldots = x^{-\beta}\prod_{j=1}^{n}\left(j-\beta+y\frac{\mathrm{d}}{\mathrm{d}y}\right)\left(\mathcal{T}_1^{0,n-\beta}f\right)(y).\end{aligned} \tag{29}$$

Now, by using the right-hand–sided Erdèlyi-Kober fractional differential operator $(\mathcal{Q}_{\varrho}^{\vartheta,\beta})$ (24) in (29), we have

$$\frac{\partial^n}{\partial x^n}\left(x^{n-\beta}\left(\mathcal{T}_1^{0,n-\beta}f\right)(y)\right) = x^{-\beta}(\mathcal{Q}_1^{-\beta,\beta}f)(y). \tag{30}$$

Therefore, by using (30) in (26), we obtain

$$\frac{\partial^\beta H}{\partial x^\beta} = t^{\frac{\gamma(\beta-3)}{\beta+3}}x^{-\beta}\left(\mathcal{Q}_1^{-\beta,\beta}f\right)(y). \tag{31}$$

Similarly, we may have

$$\frac{\partial^\beta U}{\partial x^\beta} = t^{\frac{2\gamma(\beta-3)}{\beta+3}}x^{-\beta}\left(\mathcal{Q}_1^{-\beta,\beta}h\right)(y). \tag{32}$$

Also, in a similar way, the fractional partial derivatives $\frac{\partial^\gamma H}{\partial t^\gamma}$ and $\frac{\partial^\gamma U}{\partial t^\gamma}$ are obtained of order $\gamma > 0$ as follows:

$$\begin{aligned}\frac{\partial^\gamma H}{\partial t^\gamma} &= t^{\frac{-6\gamma}{\beta+3}}\left(\mathcal{P}_{\frac{\beta+3}{\gamma}}^{1-\frac{6\gamma}{\beta+3},\gamma}f\right)(y),\\ \frac{\partial^\gamma U}{\partial t^\gamma} &= t^{\frac{\gamma(\beta-9)}{\beta+3}}\left(\mathcal{P}_{\frac{\beta+3}{\gamma}}^{1+\frac{\gamma(\beta-9)}{\beta+3},\gamma}h\right)(y),\end{aligned} \tag{33}$$

where $(\mathcal{P}_{\varrho}^{\vartheta,\gamma})$ is the left-hand–sided Erdèlyi-Kober fractional differential operator given by (23).

Hence, by using (31), (32) and (33), the space-time fractional variant Boussinesq system (1), is reduced to the NLFODE for $0 < \gamma$, $\beta < 1$ written as

$$\begin{aligned}&\left(\mathcal{P}_{\frac{\beta+3}{\gamma}}^{1-\frac{6\gamma}{\beta+3},\gamma}f\right)(y) + y^{-\beta}\left(\left(\mathcal{Q}_1^{-\beta,\beta}h\right)(y) + f(y)\left(\mathcal{Q}_1^{-\beta,\beta}f\right)(y)\right) = 0,\\ &\left(\mathcal{P}_{\frac{\beta+3}{\gamma}}^{1+\frac{\gamma(\beta-9)}{\beta+3},\gamma}h\right)(y) + y^{-\beta}\left(h(y)\left(\mathcal{Q}_1^{-\beta,\beta}f\right)(y) + f(y)\left(\mathcal{Q}_1^{-\beta,\beta}h\right)(y)\right) + f'''(y) = 0.\end{aligned} \tag{34}$$

4 Explicit power series solutions

In this section, the power series method [27,28] is applied to obtain the explicit power series solution of system (1).

Let

$$f(y) = \sum_{n=0}^{\infty}a_ny^n, \qquad h(y) = \sum_{n=0}^{\infty}b_ny^n, \tag{35}$$

where, a_n and b_n for $n = 0, 1, 2, \ldots$, are coefficients to be determined.

From (35), we have

$$\begin{aligned}f'(y) &= \sum_{n=0}^{\infty}na_ny^{n-1}, \qquad f''(y) = \sum_{n=0}^{\infty}n(n-1)a_ny^{n-2},\\ f'''(y) &= \sum_{n=0}^{\infty}n(n-1)(n-2)a_ny^{n-3}.\end{aligned} \tag{36}$$

Substituting the values from (35), (36) in (22), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{\varGamma\left(1+\frac{\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\varGamma\left(1-\frac{6\gamma}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right)a_n y^n + y^{-\beta}\left(\sum_{n=0}^{\infty}\left(\frac{\varGamma(1+n)}{\varGamma(1-\beta+n)}\right)b_n y^n+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{\varGamma(1+k)}{\varGamma(1-\beta+k)}\right)a_k a_{n-k}y^n\right)=0,\\
&\sum_{n=0}^{\infty}\left(\frac{\varGamma\left(1+\frac{2\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\varGamma\left(1+\frac{\gamma(\beta-9)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right)b_n y^n + y^{-\beta}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{\varGamma(1+n-k)}{\varGamma(1-\beta+n-k)}\right)(b_k a_{n-k}+a_k b_{n-k})y^n\right)\\
&+\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)a_{n+3}y^n=0.
\end{aligned}
\tag{37}
$$

Now comparing coefficient for $n=0$, in (37), we obtained the following values:

$$
\begin{aligned}
a_3&=-\frac{1}{6}\left\{\left(\frac{\varGamma\left(1+\frac{2\gamma(\beta-3)}{\beta+3}\right)}{\varGamma\left(1+\frac{\gamma(\beta-9)}{\beta+3}\right)}\right)b_0+2y^{-\beta}\left(\frac{1}{\varGamma(1-\beta)}\right)a_0 b_0\right\},\\
b_0&=-\left\{\left(\frac{\varGamma(1-\beta)\varGamma\left(1+\frac{\gamma(\beta-3)}{\beta+3}\right)}{\varGamma\left(1-\frac{6\gamma}{\beta+3}\right)}\right)a_0 y^{\beta}+a_0^2\right\}.
\end{aligned}
\tag{38}
$$

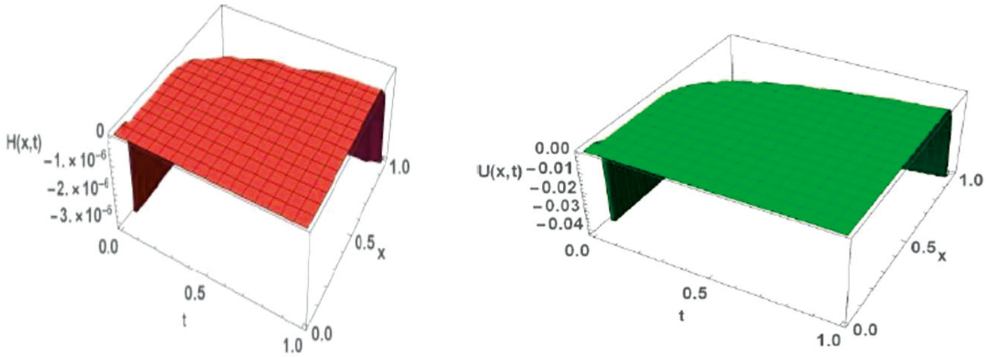
Now comparing coefficient for $n\geq 1$, in (37), we have the following recursion relation:

$$
\begin{aligned}
a_{n+3}&=-\frac{1}{(n+1)(n+2)(n+3)}\left\{\left(\frac{\varGamma\left(1+\frac{2\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\varGamma\left(1+\frac{\gamma(\beta-9)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right)b_n\right.\\
&\qquad\left.+\,y^{-\beta}\left(\sum_{k=0}^{n}\left(\frac{\varGamma(1+n-k)}{\varGamma(1+n-k-\beta)}\right)(a_{n-k}b_k+a_k b_{n-k})\right)\right\},\\
b_n&=-\frac{\varGamma(1+n-\beta)}{\varGamma(1+n)}\left\{\left(\frac{\varGamma\left(1+\frac{\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\varGamma\left(1-\frac{6\gamma}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right)a_n y^{\beta}+\sum_{k=0}^{n}\left(\frac{\varGamma(1+k)}{\varGamma(1+k-\beta)}\right)a_k a_{n-k}\right\}.
\end{aligned}
\tag{39}
$$

In view of (39), all coefficients $a_n (n\geq 3)$ and b_n $(n\geq 0)$ of the power series (35) can be found by choosing arbitrary constants a_i $(i=0,1,2)$.

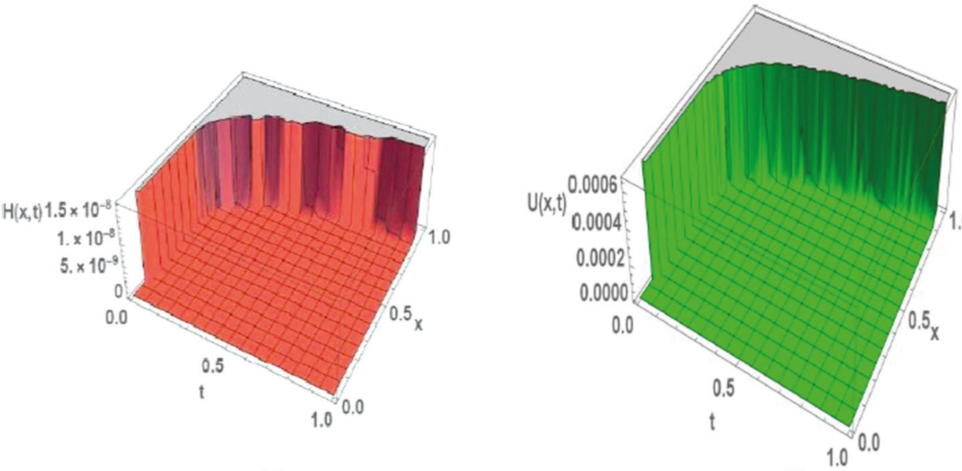
Hence, (1) have the exact power series solution having coefficients depending on (39). Thus, the power series solution for (35) is described as

$$
\begin{aligned}
f(y)&=a_0+a_1 y+a_2 y^2-\frac{1}{6}\left(\left(\frac{\varGamma\left(1+\frac{2\gamma(\beta-3)}{\beta+3}\right)}{\varGamma\left(1+\frac{\gamma(\beta-9)}{\beta+3}\right)}\right)b_0+2y^{-\beta}\left(\frac{1}{\varGamma(1-\beta)}\right)a_0 b_0\right)y^3+\sum_{n=1}^{\infty}-\frac{1}{(n+1)(n+2)(n+3)}\\
&\quad\times\left\{\left(\frac{\varGamma\left(1+\frac{2\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\varGamma\left(1+\frac{\gamma(\beta-9)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right)b_n+y^{-\beta}\left(\sum_{k=0}^{n}\left(\frac{\varGamma(1+n-k)}{\varGamma(1+n-k-\beta)}\right)(a_{n-k}b_k+a_k b_{n-k})\right)\right\}y^{n+3},\\
h(y)&=b_0+\sum_{n=1}^{\infty}-\frac{\varGamma(1+n-\beta)}{\varGamma(1+n)}\left\{\left(\frac{\varGamma\left(1+\frac{\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\varGamma\left(1-\frac{6\gamma}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right)a_n y^{\beta}+\sum_{k=0}^{n}\left(\frac{\varGamma(1+k)}{\varGamma(1+k-\beta)}\right)a_k a_{n-k}\right\}y^n.
\end{aligned}
\tag{40}
$$



(a) (b)

Fig. 1. Panels (a) and (b) show the behavior of the exact solution of $H(x,t)$ and $U(x,t)$, respectively, in (41) w.r.t x and t, when $\gamma = .25$, $\beta = .75$, $a_0 = a_1 = a_2 = 1$, $n = 45$.



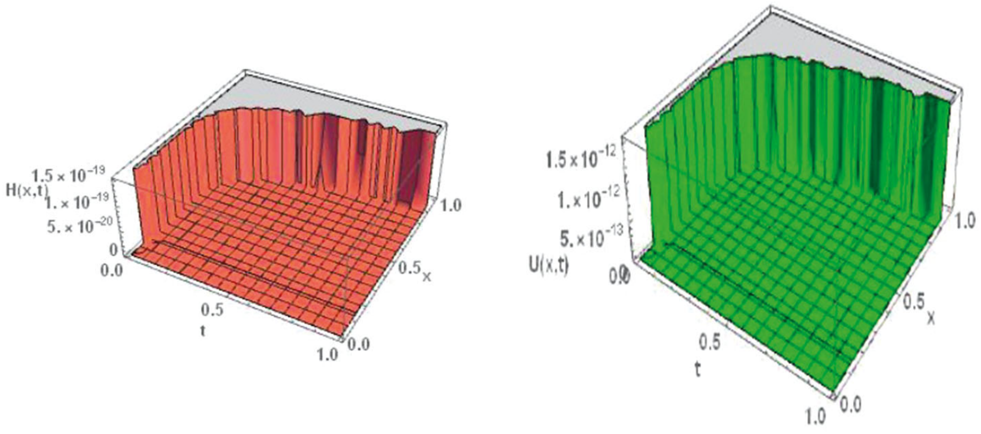
(a) (b)

Fig. 2. Panels (a) and (b) show the behavior of the exact solution of $H(x,t)$ and $U(x,t)$, respectively, in (41) w.r.t x and t, when $\gamma = .25$, $\beta = .75$, $a_0 = a_1 = a_2 = 1$, $n = 115$.

Therefore, the required explicit power series solution of (1) is

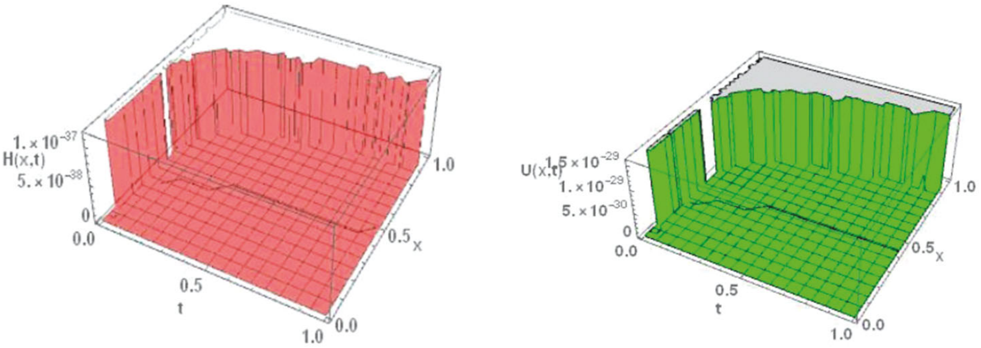
$$
\begin{aligned}
H(x,t) &= a_0 t^{\frac{\gamma(\beta-3)}{\beta+3}} + a_1 x t^{\frac{\gamma(\beta-5)}{\beta+3}} + a_2 x^2 t^{\frac{\gamma(\beta-7)}{\beta+3}} - \frac{1}{6}\left(\left(\frac{\Gamma\left(1+\frac{2\gamma(\beta-3)}{\beta+3}\right)}{\Gamma\left(1+\frac{\gamma(\beta-9)}{\beta+3}\right)}\right) b_0 + 2x^{-\beta} t^{(\beta)\frac{2\gamma}{\beta+3}} \frac{1}{\Gamma(1-\beta)} a_0 b_0\right) x^3 t^{-3\frac{2\gamma}{\beta+3}} \\
&\quad + \sum_{n=1}^{\infty} -\frac{1}{(n+1)(n+2)(n+3)} \left\{\left(\frac{\Gamma\left(1+\frac{2\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\Gamma\left(1+\frac{\gamma(\beta-9)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right) b_n x^{n+3} t^{\frac{\gamma(\beta-2n-9)}{\beta+3}} + \left(\sum_{k=0}^{n} \frac{\Gamma(1+n-k)}{\Gamma(1+n-k-\beta)}\right.\right. \\
&\quad \left.\left. \times\ (a_{n-k} b_k + a_k b_{n-k})\right) x^{-\beta+n+3} t^{\frac{\gamma(3\beta-2n-9)}{\beta+3}} \right\}, \\
U(x,t) &= -\left(\frac{\Gamma(1-\beta)\Gamma\left(1+\frac{\gamma(\beta-3)}{\beta+3}\right)}{\Gamma\left(1-\frac{6\gamma}{\beta+3}\right)}\right) a_0 x^{\beta} t^{\frac{-6\gamma}{\beta+3}} + a_0^2 t^{\frac{2\gamma(\beta-3)}{\beta+3}} + \sum_{n=1}^{\infty} -\frac{\Gamma(1+n-\beta)}{\Gamma(1+n)} \left\{\left(\frac{\Gamma\left(1+\frac{\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}{\Gamma\left(1-\frac{6\gamma}{\beta+3}-\frac{n\gamma}{\beta+3}\right)}\right)\right. \\
&\quad \left. \times\ a_n x^{n+\beta} t^{\frac{-2\gamma(n+3)}{\beta+3}} + \sum_{k=0}^{n} \left(\frac{\Gamma(1+k)}{\Gamma(1+k-\beta)}\right) a_k a_{n-k} x^n t^{\frac{2\gamma(\beta-n-3)}{\beta+3}} \right\}. \qquad (41)
\end{aligned}
$$

From figs. 1–4, one can see that the terms of the power series solution are decreasing as the value of n increases.



(a) (b)

Fig. 3. Panels (a) and (b) show the behavior of the exact solution of $H(x,t)$ and $U(x,t)$, respectively, in (41) w.r.t x and t, when $\gamma = .25$, $\beta = .75$, $a_0 = a_1 = a_2 = 1$, $n = 445$.



(a) (b)

Fig. 4. Panels (a) and (b) show the behavior of the exact solution of $H(x,t)$ and $U(x,t)$, respectively, in (41) w.r.t x and t, when $\gamma = .25$, $\beta = .75$, $a_0 = a_1 = a_2 = 1$, $n = 1000$.

5 Convergence analysis

The present section contains the analysis of convergence [24, 29] of the power series solution (41), obtained in the previous section. Now from (39), we have

$$|a_{n+3}| \leq M\left(|b_n| + \sum_{k=0}^{n} |a_{n-k}||b_k| + \sum_{k=0}^{n} |a_k||b_{n-k}|\right), \tag{42}$$

where $M = \max\left\{\left(\frac{\Gamma(1+\frac{2\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3})}{\Gamma(1+\frac{\gamma(\beta-9)}{\beta+3}-\frac{n\gamma}{\beta+3})}\right), y^{-\beta}\left(\frac{\Gamma(1+n-k)}{\Gamma(1+n-k-\beta)}\right)\right\}$, for all $k = 0, 1, 2, \ldots, n$,

$$|b_n| \leq N\left(|a_n| + \sum_{k=0}^{n} |a_k||a_{n-k}|\right), \tag{43}$$

where $N = \max\left\{\left(\frac{\Gamma(1+\frac{\gamma(\beta-3)}{\beta+3}-\frac{n\gamma}{\beta+3})}{\Gamma(1-\frac{6\gamma}{\beta+3}-\frac{n\gamma}{\beta+3})}\right) y^{\beta}, \frac{\Gamma(1+k)}{\Gamma(1+k-\beta)}\right\}$, for all $k = 0, 1, 2, \ldots, n$.

We define two power series:

$$P = P(y) = \sum_{n=0}^{\infty} p_n y^n \quad \text{and} \quad Q = Q(y) = \sum_{n=0}^{\infty} q_n y^n, \tag{44}$$

such that

$$p_i = |a_i|, \quad q_j = |b_j|, \quad \text{for } i = 0, 1, 2, 3, \text{ and } j = 0,$$

and

$$
\begin{aligned}
p_{n+3} &= M\left(q_n + \sum_{k=0}^{n} p_{n-k}q_k + \sum_{k=0}^{n} p_k q_{n-k}\right),\\
q_n &= N\left(p_n + \sum_{k=0}^{k} p_k p_{n-k}\right),
\end{aligned}
\tag{45}
$$

where $n = 0, 1, 2, 3, \ldots$. Then it can be easily seen that

$$
|a_n| \leq p_n, \quad |b_n| \leq q_n, \quad n = 0, 1, 2, \ldots. \tag{46}
$$

Hence, the series

$$
P = P(y) = \sum_{n=0}^{\infty} p_n y^n, \qquad Q = Q(y) = \sum_{n=0}^{\infty} q_n y^n
$$

is the majorant series of the series $f(y)$ and $g(y)$ in (35), respectively. Further, it is demonstrated that the series $P = P(y)$ and $Q = Q(y)$ are convergent:

$$
\begin{aligned}
P &= p_0 + p_1 y + p_2 y^2 + p_3 y^3 + \sum_{n=1}^{\infty} p_{n+3} y^{n+3}\\
&= p_0 + p_1 y + p_2 y^2 + p_3 y^3 + \sum_{n=1}^{\infty} M\left(q_n + \sum_{k=0}^{n} p_{n-k}q_k + \sum_{k=0}^{n} p_k q_{n-k}\right) y^{n+3}\\
&= p_0 + p_1 y + p_2 y^2 + p_3 y^3 + M[y^3(Q - q_0) + y^3(PQ - p_0 q_0) + y^3(PQ - p_0 q_0)],
\end{aligned}
$$

and

$$
\begin{aligned}
Q = q_0 + \sum_{n=1}^{\infty} q_n y^n &= q_0 + \sum_{n=1}^{\infty} N\left(p_n + \sum_{k=0}^{n} p_k p_{n-k}\right)\\
&= q_0 + N\left[(P - p_0) + (P^2 - p_0^2)\right].
\end{aligned}
$$

Consider the implicit functional system with respect to the independent variable y,

$$
\begin{aligned}
F(y, P, Q) &= P - p_0 - p_1 y - p_2 y^2 - p_3 y^3 - M[y^3(Q - q_0) + 2y^3(PQ - p_0 q_0)],\\
G(y, P, Q) &= Q - q_0 - N[(P - p_0) + (P^2 - p_0^2)].
\end{aligned}
$$

Since F, G are analytic in the neighbourhood of $(0, p_0, q_0)$ and $F(0, p_0, q_0) = 0$, $G(0, p_0, q_0) = 0$.

Furthermore, the Jacobian determinant

$$
\left.\frac{\partial(F, G)}{\partial(P, Q)}\right|_{(0, p_0, q_0)} = 1 \neq 0.
$$

Therefore, using the implicit function theorem [30], the series $P = P(y)$ and $Q = Q(y)$ are analytic in a neighbourhood of the point $(0, p_0, q_0)$. This implies that the two power series (35) are convergent and converge in a neighbourhood of the point $(0, p_0, q_0)$.

6 Nonlinear self-adjointness and conservation laws

In this section, the nonlinear self-adjointness [13,14,16,31] of (1), will be discussed in order to construct conservation laws of (1) by using the new conservation theorem.

The conservation laws for (1) are introduced as

$$
D_t(C^t) + D_x(C^x) = 0, \tag{47}
$$

where $C^t(x, t, H, U)$ and $C^x(x, t, H, U)$ are conserved vectors of (1) and generate a vector field. The formal Lagrangian for the system (1) is given as

$$
\mathcal{L} = u\left(\frac{\partial^\gamma H}{\partial t^\gamma} + H\frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^\beta U}{\partial x^\beta}\right) + v\left(\frac{\partial^\gamma U}{\partial t^\gamma} + H\frac{\partial^\beta U}{\partial x^\beta} + U\frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^3 H}{\partial x^3}\right), \tag{48}
$$

where u and v are new auxiliary-dependent variables.

The adjoint equations are defined as

$$F_j^* \equiv \frac{\delta \mathcal{L}}{\delta H^j} = 0, \quad j = 1, 2. \tag{49}$$

Here, $\frac{\delta}{\delta H^j}$ represent the Euler-Lagrange operators given by

$$\frac{\delta}{\delta H^j} = \frac{\partial}{\partial H^j} + (D_t^\gamma)^* \frac{\partial}{\partial (D_t^\gamma H^j)} + (D_x^\beta)^* \frac{\partial}{\partial (D_x^\beta H^j)} + \sum_{k=1}^{\infty} (-1)^k D_{i_1} D_{i_2}, \ldots, D_{i_k} \frac{\partial}{\partial (H^j)_{i_1, i_2, \ldots, i_k}}, \tag{50}$$

where D_{i_k} is the total derivative operator with respect to the i_k-th variable. Also $(D_t^\gamma)^*$ and $(D_x^\beta)^*$ are the adjoint operators [25,32] of the Riemann-Liouville fractional derivative operators D_t^γ and D_x^β, respectively.

From (48) and (49), the adjoint equations are obtained as

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta H} &= F_1^* = (D_t^\gamma)^* u + (D_x^\beta)^*(uH) + (D_x^\beta)^*(vU) + u \partial_x^\beta H + v \partial_x^\beta U - v_{xxx} = 0, \\ \frac{\delta \mathcal{L}}{\delta U} &= F_2^* = (D_t^\gamma)^* v + (D_x^\beta)^* u + (D_x^\beta)^*(vH) + u \partial_x^\beta H = 0. \end{aligned} \tag{51}$$

The system (1) is nonlinear self-adjoint if by substituting the value of new dependent variables

$$u = \varphi(x, t, H, U), \qquad v = \psi(x, t, H, U), \tag{52}$$

eq. (51) must satisfy, with at least one of the introduced new dependent variables as non zero. Now, the derivatives of $v = \psi(x, t, H, U)$ with respect to x are

$$\begin{aligned} v_x &= \psi_x + \psi_H H_x + \psi_U U_x, \\ v_{xx} &= \psi_{xx} + 2\psi_{xH} H_x + 2\psi_{xU} U_x + \psi_{HH} H_x^2 + 2\psi_{HU} H_x U_x + \psi_{UU} U_x^2 + \psi_H H_{xx} + \psi_U U_{xx}, \\ v_{xxx} &= \psi_{xxx} + 6\psi_{xHU} H_x U_x + 3\psi_{HHU} H_x^2 U_x + 3\psi_{HH} H_x U_{xx} + 3\psi_{HUU} H_x U_x^2 + 3\psi_{xH} H_{xx} \\ &\quad + 3\psi_{HU}(H_x U_{xx} + U_x H_{xx}) + 3\psi_{UU} U_x U_{xx} + 3\psi_{xxU} U_x + 3\psi_{xxH} H_x + 3\psi_{xU} U_{xx} \\ &\quad + \psi_H H_{xxx} + \psi_U U_{xxx} + 3\psi_{xHH} H_x^2 + 3\psi_{xUU} U_x^2 + \psi_{HHH} H_x^3 + \psi_{UUU} U_x^3. \end{aligned} \tag{53}$$

Thus, the nonlinear self adjointness conditions are obtained as

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta H} &= \lambda_1 \left(\frac{\partial^\gamma H}{\partial t^\gamma} + H \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^\beta U}{\partial x^\beta} \right) + \lambda_2 \left(\frac{\partial^\gamma U}{\partial t^\gamma} + H \frac{\partial^\beta U}{\partial x^\beta} + U \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^3 H}{\partial x^3} \right), \\ \frac{\delta \mathcal{L}}{\delta U} &= \lambda_3 \left(\frac{\partial^\gamma H}{\partial t^\gamma} + H \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^\beta U}{\partial x^\beta} \right) + \lambda_4 \left(\frac{\partial^\gamma U}{\partial t^\gamma} + H \frac{\partial^\beta U}{\partial x^\beta} + U \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^3 H}{\partial x^3} \right), \end{aligned} \tag{54}$$

where λ_i $(i = 1, 2, 3, 4)$ are undetermined coefficients.

Therefore, we have

$$\begin{aligned} &(D_t^\gamma)^* \varphi + (D_x^\beta)^*(\varphi H) + (D_x^\beta)^*(\psi U) + \varphi \partial_x^\beta H + \psi \partial_x^\beta U - \psi_{xxx} - 6\psi_{xHU} H_x U_x - 3\psi_{HH} H_x U_{xx} \\ &- 3\psi_{xxU} U_x - 3\psi_{xH} H_{xx} - 3\psi_{HHU} H_x^2 U_x - 3\psi_{HUU} H_x U_x^2 - 3\psi_{xxH} H_x - 3\psi_{HU}(H_x U_{xx} - U_x H_{xx}) \\ &- 3\psi_{UU} U_x U_{xx} - 3\psi_{xU} U_{xx} - \psi_H H_{xxx} - \psi_U U_{xxx} - 3\psi_{xHH} H_x^2 - 3\psi_{xUU} U_x^2 - \psi_{HHH} H_x^3 - \psi_{UUU} U_x^3 = \\ &\lambda_1 \left(\frac{\partial^\gamma H}{\partial t^\gamma} + H \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^\beta U}{\partial x^\beta} \right) + \lambda_2 \left(\frac{\partial^\gamma U}{\partial t^\gamma} + H \frac{\partial^\beta U}{\partial x^\beta} + U \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^3 H}{\partial x^3} \right), \\ &(D_t^\gamma)^* \psi + (D_x^\beta)^* \varphi + (D_x^\beta)^*(\psi H) + \varphi \partial_x^\beta H = \\ &\lambda_3 \left(\frac{\partial^\gamma H}{\partial t^\gamma} + H \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^\beta U}{\partial x^\beta} \right) + \lambda_4 \left(\frac{\partial^\gamma U}{\partial t^\gamma} + H \frac{\partial^\beta U}{\partial x^\beta} + U \frac{\partial^\beta H}{\partial x^\beta} + \frac{\partial^3 H}{\partial x^3} \right). \end{aligned} \tag{55}$$

Equating the coefficients of various powers of H and U and their partial derivatives in both sides of (55), then solving them simultaneously, we get $\lambda_i = 0$, $i = 1, 2, 3, 4$. and

$$\begin{aligned} \varphi &= \varphi(x, t, H, U), \\ \psi &= x^2 A_1(t) + x A_2(t) + A_3(t), \end{aligned} \tag{56}$$

where $A_i(t)$, $(i = 1, 2, 3)$ are arbitrary functions of t.

With the help of the new conservation theorem, the Noether operators to construct conservation laws of systems FPDEs are given as follows.

The Lie characteristic functions W^1 and W^2 corresponding to symmetry generators are defined by

$$W^1 = \eta - \xi H_x - \tau U_t, \qquad W^2 = \phi - \xi H_x - \tau U_x. \tag{57}$$

The fractional Noether operators [13,31] are defined as

$$C^t = \sum_{j=1}^{2}\left[\sum_{k=0}^{m-1}(-1)^k D_t^{\gamma-1-k}(W^j)D_t^k\left(\frac{\partial\mathcal{L}}{\partial(D_t^{\gamma}H_j)}\right) - (-1)^m \mathcal{J}_1\left(W^j, D_t^m\left(\frac{\partial\mathcal{L}}{\partial(D_t^{\gamma}H_j)}\right)\right)\right], \tag{58}$$

where $m = [\gamma]+1$, and W^j, $(j=1,2)$ are given by (57) and H_j, $(j=1,2)$ are dependent variables, *i.e.*, $H_1 = H$ and $H_2 = U$, also $\mathcal{J}_1(h_1,h_2)$ is the integral defined as

$$\mathcal{J}_1(h_1,h_2) = \frac{1}{\Gamma(m-\gamma)}\int_0^t\int_t^q \frac{h_1(x,s)h_2(x,r)}{(r-s)^{\gamma+1-m}}\mathrm{d}r\,\mathrm{d}s$$

and

$$C^x = \sum_{j=1}^{2}\left[\sum_{k=0}^{n-1}(-1)^k D_x^{\beta-1-k}(W^j)D_x^k\left(\frac{\partial\mathcal{L}}{\partial(D_x^{\beta}H_j)}\right) - (-1)^n \mathcal{J}_2\left(W^j, D_x^n\left(\frac{\partial\mathcal{L}}{\partial(D_x^{\beta}H_j)}\right)\right)\right], \tag{59}$$

where $n = [\beta]+1$, and W^j, $(j=1,2)$ given by (57) and H_j, $(j=1,2)$ are dependent variables, also $\mathcal{J}_2(h_1,h_2)$ is the integral defined as

$$\mathcal{J}_2(h_1,h_2) = \frac{1}{\Gamma(n-\beta)}\int_0^x\int_x^p \frac{h_1(s,t)h_2(r,t)}{(r-s)^{\beta+1-n}}\mathrm{d}r\,\mathrm{d}s,$$

for any two functions $h_1(x,t)$ and $h_2(x,t)$.

By using (57) and symmetry generator (17), the Lie characteristic functions are

$$\begin{aligned} W^1 &= \frac{(\beta-3)}{2}H - xH_x - \frac{(3+\beta)}{2\gamma}tH_t,\\ W^2 &= (\beta-3)U - xU_x - \frac{(3+\beta)}{2\gamma}tU_t. \end{aligned} \tag{60}$$

Now, usingthe basic definitions presented above and the adjoint variables (56), the conserved vectors of system (1), are obtained as follows.

Considering the following cases:

Case 1. For $0<\gamma<1$, the t-component of the conserved vector is obtained as

$$C^t = I^{1-\gamma}(W^1)\varphi + I^{1-\gamma}(W^2)\psi + \mathcal{J}_1(W^1, D_t\varphi) + \mathcal{J}_1(W^2,\psi_t).$$

Case 2. For $1<\gamma<2$, the t-component of the conserved vector is as

$$C^t = D^{\gamma-1}(W^1)\varphi + D^{\gamma-1}(W^2)\psi - I^{2-\gamma}(W^1)\varphi - I^{2-\gamma}(W^2)\psi x + \mathcal{J}_1(W^1, D_t^2\varphi) + \mathcal{J}_1(W^2,\psi_{tt}).$$

Case 3. For $0<\beta<1$, we have the value of $\psi = a$ (constant), and the x- component of the conserved vector is as

$$C^x = I_x^{1-\beta}(W^1)(H\varphi + aW) + I_x^{1-\beta}(W^2)(\varphi + aH) + \mathcal{J}_2(W^1, D_x(H\varphi + aU)) + \mathcal{J}_2(W^2, D_x(\varphi + aH)).$$

Case 4. For $1<\beta<2$, we have the value of $\psi = a_1x + a_2$, where a_1, a_2 are arbitrary constants and the x-component of the conserved vector is obtained as

$$\begin{aligned} C^x &= D_x^{\beta-1}(W^1)(H\varphi + (a_1x+a_2)U) + D_x^{\beta-1}(W^2)(\varphi + (a_1x+a_2)H) - I_x^{2-\beta}(W^1)D_x(H\varphi + (a_1x+a_2)U)\\ &\quad - I_x^{2-\beta}(W^2)D_x(\varphi + (a_1x+a_2)H) - \mathcal{J}_2(W^1, D_x^2(H\varphi + (a_1x+a_2)U)) - \mathcal{J}_2(W^2, D_x^2(\varphi + (a_1x+a_2)H)). \end{aligned}$$

7 Residual power series method (RPSM)

In this section we will construct the residual power series (RPS) solution [21, 33] of (1). Let us consider system (1) subject to the initial condition (at $t = 0$)

$$\begin{aligned} H(x,0) &= h(x), \\ U(x,0) &= \phi(x). \end{aligned} \tag{61}$$

Suppose that the solution of (1) is expressed in the form of the fractional power series expansion about initial point $t = 0$ as given below

$$H(x,t) = \sum_{k=0}^{\infty} h_k(x) \frac{t^{k\gamma}}{\Gamma(k\gamma+1)}\,, \tag{62}$$

$$U(x,t) = \sum_{k=0}^{\infty} \phi_K(x) \frac{t^k\gamma}{\Gamma(k\gamma+1)}\,, \quad x \in I,\ t > 0. \tag{63}$$

The analytic approximate solution for the system (1) with initial condition (61) is in the form of an infinite fractional power series given by RPSM.

In order to obtain numerical values from these series, $H_m(x,t)$ and $U_m(x,t)$ denote the m-th truncated series of $H(x,t)$ and $U(x,t)$ respectively. That is,

$$H_m(x,t) = \sum_{k=0}^{m} h_k(x) \frac{t^{k\gamma}}{\Gamma(k\gamma+1)}\,, \tag{64}$$

$$U_m(x,t) = \sum_{k=0}^{m} \phi_K(x) \frac{t^k\gamma}{\Gamma(k\gamma+1)}\,, \quad x \in I,\ t > 0. \tag{65}$$

For $m = 0$ by using the initial condition, the 0-th residual power series approximate solutions of $H(x,t)$ and $U(x,t)$ are written in the following form:

$$H_0(x,t) = h_0(x) = H(x,0) = h(x), \tag{66}$$

$$U_0(x,t) = \phi_0(x) = U(x,0) = \phi(x). \tag{67}$$

Therefore, (64) and (65) becomes

$$H_m(x,t) = h(x) + \sum_{k=1}^{m} h_k(x) \frac{t^{k\gamma}}{\Gamma(k\gamma+1)}\,, \tag{68}$$

$$U_m(x,t) = \phi(x) + \sum_{k=1}^{m} \phi_K(x) \frac{t^k\gamma}{\Gamma(k\gamma+1)}\,, \quad x \in I,\ t > 0. \tag{69}$$

In this way, the m-th residual power series approximate solution $H_m(x,t)$ and $U_m(x,t)$ can be obtained if $h_k(x)$ and $\phi_k(x)$ are known for $k = 1, 2, 3, \ldots, m$.

Now let us define the residual function for (1) and (61) as follows:

$$\begin{aligned} Res_H(x,t) &= \partial_t^\gamma H + H \partial_x^\beta H + \partial_x^\beta U, \\ Res_U(x,t) &= \partial_t^\gamma U + H \partial_x^\beta U + U \partial_x^\beta H + \partial_x^3 H. \end{aligned} \tag{70}$$

Also, the m-th residual function can be expressed as

$$\begin{aligned} Res_{H,m}(x,t) &= \partial_t^\gamma H_m + H_m \partial_x^\beta H_m + \partial_x^\beta U_m, \\ Res_{U,m}(x,t) &= \partial_t^\gamma U_m + H_m \partial_x^\beta U_m + U_m \partial_x^\beta H_m + \partial_x^3 H_m, \quad m = 1, 2, 3, \ldots. \end{aligned} \tag{71}$$

Here we state some important results of $Res_{H,m}(x,t)$ and $Res_{U,m}(x,t)$, which are essential in the residual power solution:

1)

$$Res_H(x,t) = 0, \qquad Res_U(x,t) = 0; \tag{72}$$

2)

$$\begin{aligned}\lim_{m\longrightarrow\infty} R_{H,m}(x,t) &= R_H(x,t),\\ \lim_{m\longrightarrow\infty} R_{U,m}(x,t) &= R_U(x,t);\end{aligned}\tag{73}$$

3)

$$\begin{aligned}D_t^{r\gamma}R_H(x,0) &= D_t^{r\gamma}R_{H,m}(x,0)=0,\\ D_t^{r\gamma}R_U(x,0) &= D_t^{r\gamma}R_{U,m}(x,0)=0,\quad r=0,1,2,\ldots,m.\end{aligned}\tag{74}$$

Substituting the m-th truncated series (68) and (69) of $H(x,t)$ and $U(x,t)$, respectively, into (71) and calculating the fractional derivative $D_t^{(m-1)\gamma}$ of $R_{H,m}(x,t)$ and $R_{U,m}(x,t)$, for $m=1,2,3,\ldots$. at $t=0$ together with (74), we obtain the following algebraic system

$$\begin{aligned}&D_t^{(m-1)\gamma}Res_{H,m}(x,0)=0,\\ &D_t^{(m-1)\gamma}Res_{U,m}(x,0)=0,\quad 0<\gamma\le 1,\quad m=1,2,3,\ldots.\end{aligned}\tag{75}$$

The values of $h_m(x)$ and $\phi_m(x)$ are obtained by solving system (75). Therefore, the m-th residual power series approximate solution is derived.

In the below discussion, we determined in detail the 1st and 2nd residual power series approximate solution.

For $m=1$, the first RPS solution can be written in the form of

$$H_1(x,t)=h(x)+h_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\,,\tag{76}$$

$$U_1(x,t)=\phi(x)+\phi_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\,.\tag{77}$$

The 1st residual function can be written as follows:

$$\begin{aligned}Res_{H,1}(x,t)&=\partial_t^\gamma H_1+H_1\partial_x^\beta H_1+\partial_x^\beta U_1,\\ Res_{U,1}(x,t)&=\partial_t^\gamma U_1+H_1\partial_x^\beta U_1+U_1\partial_x^\beta H_1+\partial_x^3 H_1.\end{aligned}\tag{78}$$

Inserting (76) and (77) into (78) at $t=0$, we obtain

$$\begin{aligned}Res_{H,1}(x,t)=\;&\partial_t^\gamma\left(h(x)+h_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)+\left(h(x)+h_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)\partial_x^\beta\left(h(x)+h_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)\\ &+\partial_x^\beta\left(\phi(x)+\phi_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right),\end{aligned}\tag{79}$$

$$\begin{aligned}Res_{U,1}(x,t)=\;&\partial_t^\gamma\left(\phi(x)+\phi_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)+\left(h(x)+h_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)\partial_x^\beta\left(\phi(x)+\phi_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)\\ &+\left(\phi(x)+\phi_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)\partial_x^\beta\left(h(x)+h_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right)+\partial_x^3\left(h(x)+h_1(x)\frac{t^\gamma}{\Gamma(\gamma+1)}\right).\end{aligned}\tag{80}$$

From (79), (80) and (75), we have

$$h_1(x)=-\left(h(x)\partial_x^\beta f\left(x+\partial_x^\beta\phi(x)\right)\right),\tag{81}$$

$$\phi_1(x)=-\left(\phi(x)\partial_x^\beta h(x)+h(x)\partial_x^\beta\phi(x)+\partial_x^3\phi(x)\right).\tag{82}$$

Hence, the 1st approximate solution of (1) can be written as

$$H_1(x,t)=h(x)-\left(h(x)\partial_x^\beta h(x)+\partial_x^\beta\phi(x)\right)\frac{t^\gamma}{\Gamma(\gamma+1)}\,,\tag{83}$$

$$U_1(x,t)=\phi(x)-\left(\phi(x)\partial_x^\beta h(x)+h(x)\partial_x^\beta\phi(x)+\partial_x^3\phi(x)\right)\frac{t^\gamma}{\Gamma(\gamma+1)}\,.\tag{84}$$

For $m = 2$, the 2nd residual power series solution can be obtained as follows:

$$H_2(x,t) = h(x) + h_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + h_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\,, \tag{85}$$

$$U_2(x,t) = \phi(x) + \phi_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \phi_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\,. \tag{86}$$

The 2nd residual function can be written as follows:

$$\begin{aligned} Res_{H,2}(x,t) &= \partial_t^{\gamma} H_2 + H_2 \partial_x^{\beta} H_2 + \partial_x^{\beta} U_2,\\ Res_{U,2}(x,t) &= \partial_t^{\gamma} U_2 + H_2 \partial_x^{\beta} U_2 + U_2 \partial_x^{\beta} H_2 + \partial_x^{3} H_2. \end{aligned} \tag{87}$$

Inserting (85) and (86) into (87) at $t = 0$, we obtain

$$\begin{aligned} Res_{H,2}(x,t) = {} & \partial_t^{\gamma}\left(h(x) + h_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + h_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right) + \left(h(x) + h_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + h_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right)\\ & \times \partial_x^{\beta}\left(h(x) + h_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + h_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right) + \partial_x^{\beta}\left(\phi(x) + \phi_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \phi_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right), \end{aligned} \tag{88}$$

$$\begin{aligned} Res_{U,2}(x,t) = {} & \partial_t^{\gamma}\left(\phi(x) + \phi_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \phi_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right) + \left(h(x) + h_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + h_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right)\\ & \times \partial_x^{\beta}\left(\phi(x) + \phi_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \phi_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right) + \left(\phi(x) + \phi_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \phi_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right)\\ & \times \partial_x^{\beta}\left(h(x) + h_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + h_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right) + \partial_x^{3}\left(h(x) + h_1(x)\frac{t^{\gamma}}{\Gamma(\gamma+1)} + h_2(x)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right). \end{aligned} \tag{89}$$

From (88), (89) and (75), we have

$$h_2(x) = -\left(h(x)\partial_x^{\beta} h_1(x) + h_1(x)\partial_x^{\beta} h(x) + \partial_x^{\beta}\phi_1(x)\right), \tag{90}$$

$$\phi_2(x) = -\left(h(x)\partial_x^{\beta}\phi_1(x) + \phi_1(x)\partial_x^{\beta} h(x) + h_1(x)\partial_x^{\beta}\phi(x) + \phi(x)\partial_x^{\beta} h_1(x) + \partial_x^{3} h_1(x)\right). \tag{91}$$

Hence, the 2nd approximate solution of (1) can be written as

$$H_2(x,t) = h(x) - \left(h(x)\partial_x^{\beta} h(x) + \partial_x^{\beta}\phi(x)\right)\frac{t^{\gamma}}{\Gamma(\gamma+1)} - \left(h(x)\partial_x^{\beta} h_1(x) + h_1(x)\partial_x^{\beta} h(x) + \partial_x^{\beta}\phi_1(x)\right)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\,, \tag{92}$$

$$\begin{aligned} U_2(x,t) = {} & \phi(x) - \left(\phi(x)\partial_x^{\beta} h(x) + h(x)\partial_x^{\beta}\phi(x) + \partial_x^{3}\phi(x)\right)\frac{t^{\gamma}}{\Gamma(\gamma+1)} - \big(h(x)\partial_x^{\beta}\phi_1(x) + \phi_1(x)\partial_x^{\beta} h(x)\\ & + h_1(x)\partial_x^{\beta}\phi(x) + \phi(x)\partial_x^{\beta} h_1(x) + \partial_x^{3} h_1(x)\big)\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\,. \end{aligned} \tag{93}$$

In the same way, we can find the remaining approximate solution of order 3, 4 and so on of system (1). Hence, we have

$$\begin{aligned} H_m(x,t) &= \sum_{k=0}^{m} h_k(x)\frac{t^{k\gamma}}{\Gamma(k\gamma+1)}\,,\\ U_m(x,t) &= \sum_{k=0}^{m} \phi_K(x)\frac{t^{k}\gamma}{\Gamma(k\gamma+1)}\,, \quad x \in I,\ t > 0. \end{aligned} \tag{94}$$

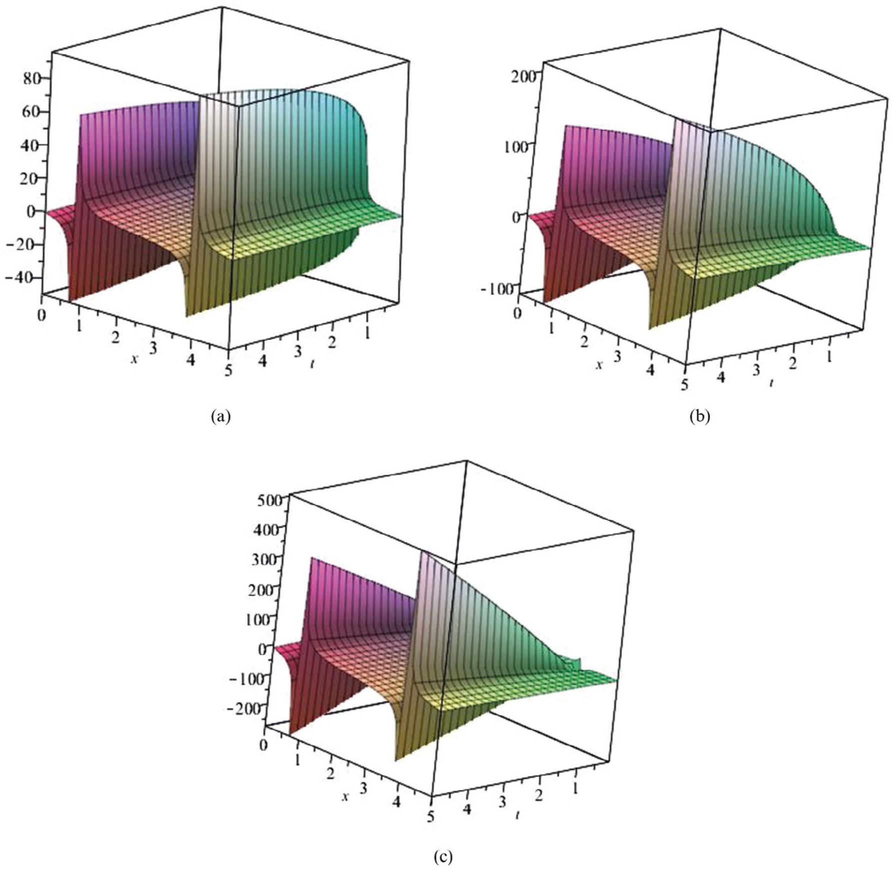


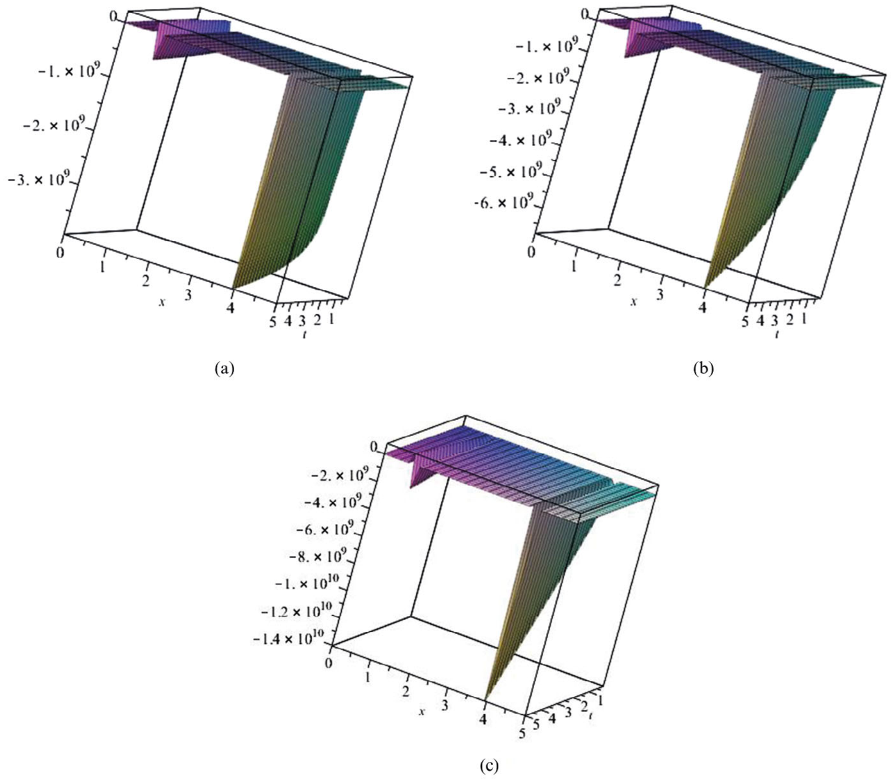
Fig. 5. Behavior of the first RPS approximate solution (83) for different values: (a) $\gamma = .1$ and $\beta = .3$; (b) $\gamma = .4$ and $\beta = .6$; (c) $\gamma = .9$ and $\beta = .9$.

By using the concept of the generalized Taylor series, the RPSM is a technique to find the analytic approximate solution of physical phenomena in the form of convergent series. By utilizing the terms of RPS approximations, higher terms will reduce the error of approximation. Here we present some graphical explanations that are reported according to the obtained first and second approximation with respect to $h(x)$ and $\phi(X)$ chosen as below:

$$\begin{aligned} h(x) &= 1 - \frac{2(\sin(x) + \cos(x))}{\sin(x) - \cos(x)}\,, \\ \phi(x) &= -2 - 2\left(\frac{\sin(x) + \cos(x)}{\sin(x) - \cos(x)}\right)^2. \end{aligned} \tag{95}$$

8 Numerical results and discussion

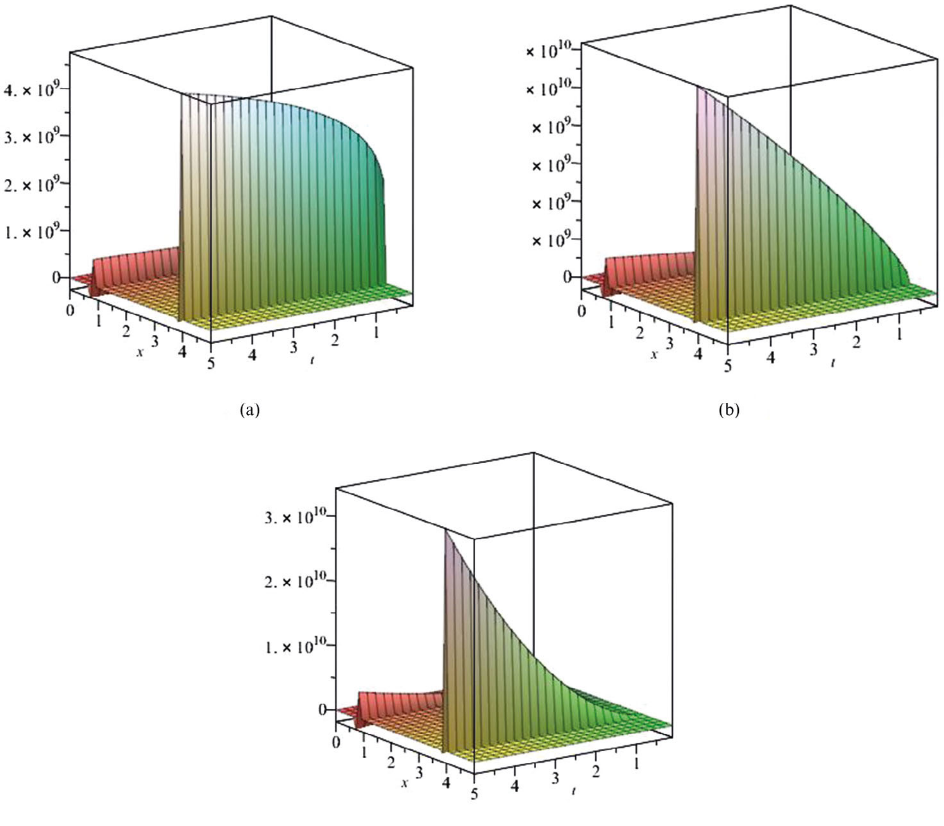
This section contains graphical explanations of the numerical simulations of RPS of system (1). Figures 5 and 6 show the behavior of the first RPS approximate solution of (83) and (84), respectively, for different values of $\gamma = .1$ and $\beta = .3$, $\gamma = .4$ and $\beta = .6$ and $\gamma = .9$ and $\beta = .9$ and $\beta = .25$, respectively. Similarly, figs. 7 and 8 show the behavior of the first RPS approximate solution of (92) and (93), respectively, for different values of $\gamma = .1$ and $\beta = .3$, $\gamma = .4$ and $\beta = .6$ and $\gamma = .9$ and $\beta = .9$, respectively. Here, one can see that if γ and β are increasing with different values, these figures are almost equal and in good agreement with one another, with respect to the correctness, and also the absolute error is very small. Therefore, it can be stated that the wave solution is smoother around the 0 as γ and β approaches 1.



(a) (b)

(c)

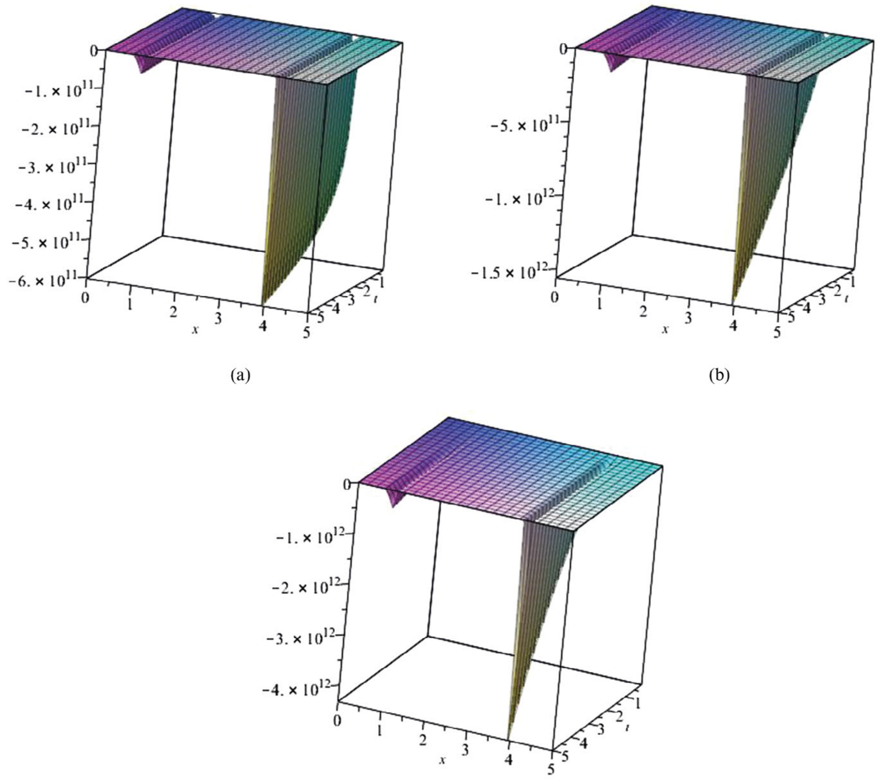
Fig. 6. Behavior of the first RPS approximate solution (84) for different values: (a) $\gamma = .1$ and $\beta = .3$; (b) $\gamma = .4$ and $\beta = .6$; (c) $\gamma = .9$ and $\beta = .9$.



(a) (b)

(c)

Fig. 7. Behavior of the first RPS approximate solution (92) for different values: (a) $\gamma = .1$ and $\beta = .3$; (b) $\gamma = .4$ and $\beta = .6$; (c) $\gamma = .9$ and $\beta = .9$.



(a) (b)

(c)

Fig. 8. Behavior of the first RPS approximate solution (93) for different values: (a) $\gamma = .1$ and $\beta = .3$; (b) $\gamma = .4$ and $\beta = .6$; (c) $\gamma = .9$ and $\beta = .9$.

9 Conclusion

In the present study, the symmetry variables and symmetry transformations of the third order space-time fractional variant Boussinesq system have been derived. We have introduced the Lie symmetry analysis for investigating systems of space-time NLFPDEs and successfully reduced the system of NLFPDEs into a system of NLFODEs. The explicit power series solution of the considered system has been derived by using the reduced NLFODEs. Also, the obtained power series solution has been analyzed for convergence. The conservation laws for the space-time fractional governing system are constructed by using the generalized Noether's theorem and new conservation theorem. Some authors have studied the same system for the integer order derivative and for the time fractional order derivative. In comparison with them, our result recovers the symmetries of the system of the integer order derivative for $\gamma = \beta = 1$ studied in ref. [20]. The approximate solutions of the governing system subject to the given initial conditions are constructed with accuracy using the RPSM, which is the modern analytical iterative technique. The graphical results demonstrated that the nearly similar and coinciding behavior of the RPS approximate solution for $\gamma = .1$ and $\beta = .3$, $\gamma = .4$ and $\beta = .6$ and $\gamma = .9$ and $\beta = .9$ in terms of accuracy.

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