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Dispersion analysis and improved F-expansion method for space–time fractional differential equations

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Abstract In this article, an improved F-expansion method with the Riccati equation is suggested for space–time fractional differential equations for exact solutions. The fractional complex transformation is used to convert the space–time fractional differential equations into ordinary differential equations. The application of the method is described by solving space–time fractional potential Yu–Toda–Sasa–Fukuyama equation, and the solutions of the equation are successfully established in terms of the hyperbolic, trigonometric and rational types of functions. The graphical analysis describes the effect of fractional orders $\alpha, \beta, \gamma, \delta$ of time and space derivatives, respectively, on the wave profile of solutions. The dispersion relation is obtained using the linear analysis, and it shows that waves follow anomalous or normal dispersion depending upon space or time fractional-order values.

Keywords Space–time fractional potential Yu–Toda–Sasa–Fukuyama equation · Improved F-expansion method · Exact solutions · Dispersion analysis

Mathematics Subject Classification 34A08 · 35R11 · 35C07 · 81U30

1 Introduction

The various physical phenomena are mathematically modelled in terms of fractional partial differential equations (FPDEs) in the areas of mathematics, physics and engineering [45,47,49,55] such as continuous-time random walk process [56], anomalous diffusion [72,73], memory term effect [14,19,49,68], modelling mechanical and electrical properties of real materials [18,49], description of rheological properties of rocks [49], power laws [68], fractals [20,74], steady heat flow [75] involving fractional space/time derivatives. In recent years, considerable research describes the way to find exact solutions of FPDEs and the most commonly used methods to solve FPDEs are the symmetry method [25,26,29,35,48,50,52,58,59,62], fractional sub-equation method [82], (G′/G)-expansion method [11], exponential function method [11], expo-

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nential rational function method [7,46], fractional complex transformation method [9], factorisation technique using travelling wave transformations [71], travelling wave solution method based on the formulation of local fractional derivatives [70,74], the first integral method [10], generalised Kudryashov method [22] and so on. In the literature, the improved F-expansion method with the Riccati equation is proposed to solve for exact travelling wave solutions of integral-order differential equations [6,30–34] but not for FPDEs. Therefore, our aim is to propose an improved F-expansion method to solve space–time FPDEs for exact solutions. In this paper, an improved F-expansion method is applied to obtain exact solutions of (3 + 1)-dimensional space–time Yu–Toda–Sasa–Fukuyama (YTSF) equation [77]. The integral version of YTSF equation is given by

$$\begin{aligned} &(-4w_t + \Phi(w)w_z)_x + 3w_{yy} = 0,\\ &\Phi = \partial_x^2 + 4w + 2w_x\partial_x^{-1}, \end{aligned} \tag{1}$$

where $w : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z \times \mathbb{R}_t \to \mathbb{R}$ and the various subscripts denote partial derivatives with respect to x, y, z or t and $\partial_x^{-1}(\cdot) = \int_{-\infty}^{\infty}(\cdot)\mathrm{d}x$. Equation (1) is (3 + 1)-dimensional generalisation of Calogero–Bogoyavlenskii–Schiff equation [57] given by

$$w_t + \Phi(w)w_z = 0, \;\; \Phi(w) = \partial_x^2 + 4w + 2w_x\partial_x^{-1}. \tag{2}$$

Using $w = u_x$ in Eq. (1), it gives (3 + 1)-dimensional potential YTSF equation [77] as follows

$$-4\,u_{xt} + u_{xxxz} + 3\,u_{yy} + 4\,u_x u_{xz} + 2\,u_{xx}u_z = 0. \tag{3}$$

In Eq. (3), the function $u(x, y, z, t)$ represents the amplitude of the relevant wave and is used to describe the physical systems such as the mobility of solitons and nonlinear waves in the fields of plasma physics, fluid dynamics, weakly dispersive media, etc. [38]. Equation (3) has been solved for solutions in terms of soliton, non-travelling wave, exact travelling wave, rogue wave, soliton solutions in Gramian, rational, lump and solitary waves [8,12,13,15,17,28,33,39–41,54,61,63,64,67,76,78–81,83]. In the literature, no reports on space–time fractional form of Eq. (3) have been found for exact solutions and dispersion analysis. The fractional analysis provides diversity in understanding the dynamics of various physical systems. The real-life application of YTSF equation is found in oceanography and waves in two-layer liquid medium and elastic medium [1,15,39,61,65,76]. The fractional-order version of Eq. (3) can be obtained using the variational principle for fractional calculus [3,44], and the proof is given in “Appendix”. This formulation is successfully implemented for deriving fractional-order differential equations for other systems reported in [3,21,24,44,53]. In the present analysis, the space–time fractional potential YTSF equation in normalised form is considered as follows

$$\begin{aligned} &-4\,\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) + \frac{\partial^{3\beta}}{\partial x^{3\beta}}\left(\frac{\partial^\delta u}{\partial z^\delta}\right) + 3\,\frac{\partial^{2\gamma} u}{\partial y^{2\gamma}}\\ &\quad + 4\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\delta u}{\partial z^\delta}\right)\right)\\ &\quad + 2\left(\frac{\partial^{2\beta} u}{\partial x^{2\beta}}\right)\left(\frac{\partial^\delta u}{\partial z^\delta}\right) = 0. \end{aligned} \tag{4}$$

where α, β, γ and δ ($0 < \alpha, \beta, \gamma, \delta < 1$) denote the fractional orders of the derivatives with respect to independent variables t, x, y and z, respectively.

Linear dispersion analysis of FPDEs [16,23] gives the relation between wave frequency and wave vectors which is further used to find phase and group velocities.

The paper is organised as follows. In Sect. 2, the description of improved F-expansion method for solving space–time fractional FPDEs is given. In Sect. 3, the application of the proposed method is given to solve Eq. (4) and solutions are obtained. The graphical analysis describes the effect of fractional orders on the wave profile of solutions. Section 4 provides the linear analysis of the equation which gives dispersion relation; further, phase and group velocities are obtained. In the last section, concluding remarks are given.

2 Detailed description of improved F-expansion method for space–time FPDEs

This section proposes the detailed algorithm of improved F-expansion method for obtaining exact travelling wave solutions of space–time FPDEs. For this, consider the definition of Riemann–Liouville (RL) fractional derivative [49] of order $\alpha > 0$, $\alpha \in \mathbb{R}^+$ with lower terminal at 0 as follows

$$D_t^\alpha f(t) = \begin{cases} \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha = n, \text{ where } n \in \mathbb{N};\\ \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-\theta)^{n-\alpha-1} f(\theta)\,\mathrm{d}\theta, & n-1 < \alpha < n,\ n \in \mathbb{N}. \end{cases} \tag{5}$$

The RL fractional derivative (5) exhibits the following property [49]

$$D^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)t^{\gamma-\alpha}}{\Gamma(\gamma-\alpha+1)},\ \gamma>-1, \tag{6}$$

where $\Gamma(\cdot)$ is the gamma function.

To apply improved F-expansion method onto the space–time FPDEs, firstly, consider the Riccati equation as follows

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta}=l+\phi^{2}(\zeta), \tag{7}$$

where l is the real parameter, and the general solutions of Riccati Eq. (7) are examined in the following cases.

Case 1 For $l<0$, the hyperbolic functions solutions of Eq. (7) are given as follows

$$\begin{aligned}\phi_1&=-\sqrt{-l}\tanh(\sqrt{-l}\,\zeta),\\ \phi_2&=-\sqrt{-l}\coth(\sqrt{-l}\,\zeta).\end{aligned} \tag{8}$$

Case 2 When $l>0$, the solutions of Eq. (7) can be written in terms of trigonometric functions as follows

$$\begin{aligned}\phi_3&=\sqrt{l}\tan(\sqrt{l}\,\zeta),\\ \phi_4&=-\sqrt{l}\cot(\sqrt{l}\,\zeta).\end{aligned} \tag{9}$$

Case 3 For $l=0$, the solution of Eq. (7) is given in the following rational form

$$\phi_5=-\frac{1}{\zeta}. \tag{10}$$

To find the exact solutions, the space–time FPDE is considered in the following form

$$\begin{aligned}&\Xi\left(u,\frac{\partial^{\alpha}u}{\partial t^{\alpha}},\frac{\partial^{\beta}u}{\partial x^{\beta}},\frac{\partial^{\gamma}u}{\partial y^{\gamma}},\frac{\partial^{\delta}u}{\partial z^{\delta}},\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right),\right.\\ &\left.\quad\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right),\frac{\partial^{2\beta}u}{\partial x^{2\beta}},\frac{\partial^{3\beta}}{\partial x^{3\beta}}\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right),\ldots\right)=0,\\ &\quad 0<\alpha,\beta,\gamma,\delta<1,\end{aligned} \tag{11}$$

where α, β, γ and δ are the fractional orders of the derivatives of u corresponding to variables t, x, y and z, respectively. The solution of Eq. (11) in terms of fractional complex transformation [27,37,51,60,69] is given as follows

$$\begin{aligned}&u(x,y,z,t)=U(\zeta),\\ &\zeta=\frac{k_1x^{\beta}}{\Gamma(\beta+1)}+\frac{k_2y^{\gamma}}{\Gamma(\gamma+1)}\\ &\qquad+\frac{k_3z^{\delta}}{\Gamma(\delta+1)}+\frac{k_4t^{\alpha}}{\Gamma(\alpha+1)},\end{aligned} \tag{12}$$

where U is a function of complex variable ζ and k_1, k_2, k_3, k_4 are the arbitrary constants. Using chain rule the following equations are obtained

$$\begin{aligned}&\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\sigma_tD_{\zeta}UD_t^{\alpha}\zeta=\sigma_tk_4(D_{\zeta}U)=\sigma_tk_4U',\\ &\frac{\partial^{\beta}u}{\partial x^{\beta}}=\sigma_xD_{\zeta}UD_x^{\beta}\zeta=\sigma_xk_1(D_{\zeta}U)=\sigma_xk_1U',\\ &\frac{\partial^{\gamma}u}{\partial y^{\gamma}}=\sigma_yD_{\zeta}UD_y^{\gamma}\zeta=\sigma_yk_2(D_{\zeta}U)=\sigma_yk_2U',\\ &\frac{\partial^{\delta}u}{\partial z^{\delta}}=\sigma_zD_{\zeta}UD_z^{\delta}\zeta=\sigma_zk_3(D_{\zeta}U)=\sigma_zk_3U',\\ &\vdots\end{aligned} \tag{13}$$

where prime $'$ denotes the derivative of $U(\zeta)$ with respect to ζ and $\sigma_t, \sigma_x, \sigma_y, \sigma_z$ are the fractional indexes, and without loss of generality, consider $\sigma_t=\sigma_x=\sigma_y=\sigma_z=\rho$, where ρ is the arbitrary constant [27,51]. Thus, Eq. (12) converts Eq. (11) into ordinary differential equation (ODE) in terms of U and its derivatives in the form as follows

$$\begin{aligned}&\Phi(U,\rho k_4U',\rho k_1U',\rho k_2U',\rho k_3U',\rho^2k_4k_1U'',\\ &\quad\rho^2k_1k_3U'',\rho^2k_1^2U'',\rho^4k_1^3k_3U'''',\ldots)=0.\end{aligned} \tag{14}$$

Equation (14) can also be integrated once or more number of times. It gives the constants of integration which can be equated to zero for getting the solutions.

Let us consider the travelling wave solution for Eq. (14) in the following form

$$U(\zeta)=\sum_{i=0}^{P}\mu_i(m+\phi(\zeta))^{i}+\sum_{i=1}^{P}\nu_i(m+\phi(\zeta))^{-i}, \tag{15}$$

where $\phi(\zeta)$ satisfies the Riccati Eq. (7). The μ_i, ν_i and m are the arbitrary constants and $\mu_i, \nu_i\neq 0$, simultaneously. The parameter P can be determined by using the balancing principle in which the highest order derivative term is balanced with the highest order nonlinear term from Eq. (14).

Also, from Eq. (15), the expression for U' is obtained as follows

$$
\begin{aligned}
U' = \left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(\zeta)\right)\Bigg(&\sum_{i=0}^{P}(m+\phi(\zeta))^{i-1}\, i\mu_i \\
&-\sum_{i=1}^{P}(m+\phi(\zeta))^{-i-1}\, i\nu_i\Bigg).
\end{aligned}
\tag{16}
$$

Using Eq. (7), U' reduces to

$$
\begin{aligned}
U' = \left(l+\phi^2(\zeta)\right)\Bigg(&\sum_{i=0}^{P}(m+\phi(\zeta))^{i-1}\, i\mu_i \\
&-\sum_{i=1}^{P}(m+\phi(\zeta))^{-i-1}\, i\nu_i\Bigg).
\end{aligned}
\tag{17}
$$

The expression for U'' using Riccati Eq. (7) is obtained as follows

$$
\begin{aligned}
U'' = \left(l+\phi^2(\zeta)\right)\Bigg(&2\phi(\zeta)\sum_{i=0}^{P}(m+\phi(\zeta))^{i-1}\, i\mu_i \\
&-2\phi(\zeta)\sum_{i=1}^{P}(m+\phi(\zeta))^{-i-1}\, i\nu_i \\
&+l\sum_{i=0}^{P}(i-1)(m+\phi(\zeta))^{i-2}\, i\mu_i \\
&+l\sum_{i=1}^{P}(m+\phi(\zeta))^{-i-2}\, i(i+1)\nu_i \\
&+(\phi(\zeta))^2\sum_{i=0}^{P}(i-1)(m+\phi(\zeta))^{i-2}\, i\mu_i \\
&+\phi^2(\zeta)\sum_{i=1}^{P}(m+\phi(\zeta))^{-i-2}\, i(i+1)\nu_i\Bigg).
\end{aligned}
\tag{18}
$$

Similarly, one can obtain the expressions for other higher-order derivatives of U. The back-substitution of U and its various orders of derivatives together with the value of P into Eq. (14) results in a polynomial in terms of function $\phi(\zeta)$.

Then, equating the coefficients of various powers of $\phi(\zeta)$ to zero, it gives a system of over-determining equations whose solution provides μ_i, ν_i, m and k_4 values.

The solutions of Eq. (14) can be drafted by inserting the values of μ_i, ν_i, m and k_4 into Eq. (15) with known solutions of Eq. (7). The back-substitution of solutions of Eq. (14) into Eq. (12) gives the exact travelling wave solutions of the space–time fractional Eq. (11).

3 Application of the method to solve potential YTSF Eq. (4)

In this section, the improved F-expansion method has been applied to Eq. (4) to obtain exact travelling wave solutions. The fractional complex transformation for Eq. (4) gives the following ODE after integration

$$
\left(-4k_1k_4+3k_2^2\right)U' + k_1^3k_3\rho^2U''' + 3\rho k_1^2k_3U'^2 = 0,
\tag{19}
$$

where the constant of integration is considered as zero. The homogeneous balancing between the highest order derivative term U''' and the highest order nonlinear term U'^2 in Eq. (19) gives the value of $P = 1$, and further, the solution of Eq. (19) can be written in the following form

$$
U(\zeta) = \mu_0 + \mu_1(m+\phi(\zeta)) + \nu_1(m+\phi(\zeta))^{-1}.
\tag{20}
$$

The expression for U' is obtained as follows

$$
U' = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\phi(\zeta)\right)\left(\mu_1m^2+2\mu_1m\phi(\zeta)+\mu_1\phi^2(\zeta)-\nu_1\right)}{(m+\phi(\zeta))^2}.
\tag{21}
$$

Using Eq. (7), U' reduces to

$$
U' = \frac{\left(l+\phi^2(\zeta)\right)\left(\mu_1m^2+2\mu_1m\phi(\zeta)+\mu_1\phi^2(\zeta)-\nu_1\right)}{(m+\phi(\zeta))^2}.
\tag{22}
$$

Similarly, other expressions U'' and U''' are obtained as follows

$$
\begin{aligned}
U'' = \frac{2\left(l+\phi^2(\zeta)\right)}{(m+\phi(\zeta))^3}\Big(&\phi(\zeta)\mu_1m^3+3\mu_1m^2\phi^2(\zeta) \\
&+3\mu_1m\phi^3(\zeta)+\mu_1\phi^4(\zeta) \\
&-\phi(\zeta)\nu_1m+l\nu_1\Big),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
U''' = \frac{2\left(l+\phi^2(\zeta)\right)}{(m+\phi(\zeta))^4}\Big(&4\phi(\zeta)l\nu_1m+4l\mu_1m^3\phi(\zeta) \\
&+6l\mu_1m^2\phi^2(\zeta) \\
&+4l\mu_1m\phi^3(\zeta)+3\phi^2(\zeta)\mu_1m^4 \\
&+12\phi^3(\zeta)\mu_1m^3+18\mu_1m^2\phi^4(\zeta) \\
&+12\mu_1m\phi^5(\zeta)-3\phi^2(\zeta)\nu_1m^2 \\
&-l\nu_1\phi^2(\zeta)+l\mu_1m^4+l\mu_1\phi^4(\zeta)
\end{aligned}
$$

$$- l\nu_1 m^2 - 3\,l^2\nu_1 + 3\,\mu_1\phi^6(\zeta)\Big), \tag{24}$$

Substitution of U', U'' and U''' into Eq. (19) gives a polynomial in $\phi(\zeta)$ as follows

$$\begin{aligned}
&\left(3\,\rho\,k_1{}^2k_3\mu_1{}^2 + 6\,k_1{}^3k_3\rho^2\mu_1\right)\phi^8(\zeta) \\
&\quad+ \left(12\,\rho\,k_1{}^2k_3\mu_1{}^2m + 24\,k_1{}^3k_3\rho^2\mu_1 m\right)\phi^7(\zeta) \\
&\quad+ \Big(36\,k_1{}^3k_3\rho^2\mu_1 m^2 + 8\,k_1{}^3k_3\rho^2\mu_1 l \\
&\quad+ 18\,\rho\,k_1{}^2k_3\mu_1{}^2m^2 + 3\,k_2{}^2\mu_1 - 4\,k_1k_4\,\mu_1 \\
&\quad-6\,\rho\,k_1{}^2k_3\mu_1\nu_1 + 6\,\rho\,k_1{}^2k_3\mu_1{}^2l\Big)\phi^6(\zeta) \\
&\quad+ \Big(24\,\rho\,k_1{}^2k_3\mu_1{}^2ml + 12\,k_2{}^2\mu_1 m \\
&\quad- 16\,k_1k_4\,\mu_1 m + 12\,\rho\,k_1{}^2k_3\mu_1{}^2m^3 \\
&\quad+ 24\,k_1{}^3k_3\rho^2\mu_1 m^3 + 32\,k_1{}^3k_3\rho^2\mu_1 ml \\
&\quad-12\,\rho\,k_1{}^2k_3\mu_1 m\nu_1\Big)\phi^5(\zeta) \\
&\quad+ \Big(4\,k_1k_4\,\nu_1 - 12\,\rho\,k_1{}^2k_3\mu_1\nu_1 l - 6\,\rho\,k_1{}^2k_3\mu_1 m^2\nu_1 \\
&\quad- 24\,k_1k_4\,\mu_1 m^2 + 3\,\rho\,k_1{}^2k_3\nu_1{}^2 + 3\,\rho\,k_1{}^2k_3\mu_1{}^2m^4 \\
&\quad+ 3\,\rho\,k_1{}^2k_3\mu_1{}^2l^2 + 2\,k_1{}^3k_3\rho^2\mu_1 l^2 \\
&\quad+ 36\,\rho\,k_1{}^2k_3\mu_1{}^2m^2l + 18\,k_2{}^2\mu_1 m^2 \\
&\quad- 2\,k_1{}^3k_3\rho^2\nu_1 l - 4\,k_1k_4\,\mu_1 l - 6\,k_1{}^3k_3\rho^2\nu_1 m^2 \\
&\quad- 3\,k_2{}^2\nu_1 + 6\,k_1{}^3k_3\rho^2\mu_1 m^4 \\
&\quad+48\,k_1{}^3k_3\rho^2\mu_1 m^2l + 3\,k_2{}^2\mu_1 l\Big)\phi^4(\zeta) \\
&\quad+ \Big(12\,\rho\,k_1{}^2k_3\mu_1{}^2ml^2 - 24\,\rho\,k_1{}^2k_3\mu_1 m\nu_1 l \\
&\quad- 16\,k_1k_4\,\mu_1 m^3 + 8\,k_1{}^3k_3\rho^2\mu_1 ml^2 \\
&\quad+ 12\,k_2{}^2\mu_1 ml + 32\,k_1{}^3k_3\rho^2\mu_1 m^3l + 8\,k_1k_4\,\nu_1 m \\
&\quad+ 24\,\rho\,k_1{}^2k_3\mu_1{}^2m^3l - 6\,k_2{}^2\nu_1 m \\
&\quad+12\,k_2{}^2\mu_1 m^3 - 16\,k_1k_4\,\mu_1 ml + 8\,k_1{}^3k_3\rho^2\nu_1 ml\Big)\phi^3(\zeta) \\
&\quad+ \Big(-8\,k_1{}^3k_3\rho^2\nu_1 l^2 + 6\,\rho\,k_1{}^2k_3\nu_1{}^2l + 3\,k_2{}^2\mu_1 m^4 \\
&\quad+ 18\,\rho\,k_1{}^2k_3\mu_1{}^2m^2l^2 \\
&\quad+ 12\,k_1{}^3k_3\rho^2\mu_1 m^2l^2 + 18\,k_2{}^2\mu_1 m^2l \\
&\quad- 12\,\rho\,k_1{}^2k_3\mu_1 m^2\nu_1 l + 6\,\rho\,k_1{}^2k_3\mu_1{}^2m^4l \\
&\quad+ 8\,k_1{}^3k_3\rho^2\mu_1 m^4l - 6\,\rho\,k_1{}^2k_3\mu_1\nu_1 l^2 \\
&\quad+ 4\,k_1k_4\,\nu_1 m^2 - 3\,k_2{}^2\nu_1 m^2 - 4\,k_1k_4\,\mu_1 m^4 \\
&\quad- 24\,k_1k_4\,\mu_1 m^2l - 8\,k_1{}^3k_3\rho^2\nu_1 m^2l \\
&\quad-3\,k_2{}^2\nu_1 l + 4\,k_1k_4\,\nu_1 l\Big)\phi^2(\zeta) \\
&\quad+ \Big(8\,k_1{}^3k_3\rho^2\mu_1 m^3l^2 - 12\,\rho\,k_1{}^2k_3\mu_1 m\nu_1 l^2 \\
&\quad+ 8\,k_1k_4\,\nu_1 ml - 16\,k_1k_4\,\mu_1 m^3l \\
&\quad+ 8\,k_1{}^3k_3\rho^2\nu_1 ml^2 + 12\,k_2{}^2\mu_1 m^3l \\
&\quad+12\,\rho\,k_1{}^2k_3\mu_1{}^2m^3l^2 - 6\,k_2{}^2\nu_1 ml\Big)\phi(\zeta) \\
&\quad- 3\,k_2{}^2\nu_1 m^2l - 2\,k_1{}^3k_3\rho^2\nu_1 m^2l^2 \\
&\quad- 4\,k_1k_4\,\mu_1 m^4l - 6\,\rho\,k_1{}^2k_3\mu_1 m^2\nu_1 l^2 \\
&\quad+ 3\,\rho\,k_1{}^2k_3\mu_1{}^2m^4l^2 - 6\,k_1{}^3k_3\rho^2\nu_1 l^3 \\
&\quad+ 2\,k_1{}^3k_3\rho^2\mu_1 m^4l^2 + 3\,k_2{}^2\mu_1 m^4l \\
&\quad+ 4\,k_1k_4\,\nu_1 m^2l + 3\,\rho\,k_1{}^2k_3\nu_1{}^2l^2 = 0.
\end{aligned} \tag{25}$$

Now, by equating the coefficients of all powers of $\phi^i(\zeta)$, $i = 1, 2, \ldots$, to zero, the over-determining system of equations is obtained using symbolic computational software Maple. The solutions of determining equations are given by the following expressions

$$\begin{aligned}
&k_4 = -\frac{4\,k_1{}^3k_3\rho^2\,l - 3\,k_2{}^2}{4\,k_1},\ \ \mu_1 = 0, \\
&\nu_1 = 2\,k_1\rho\,l + 2\,k_1\rho\,m^2,
\end{aligned} \tag{26}$$

and

$$k_4 = -\frac{4\,k_1{}^3k_3\rho^2\,l - 3\,k_2{}^2}{4\,k_1},\ \ \mu_1 = -2k_1\rho,\ \ \nu_1 = 0. \tag{27}$$

Using the above set of values for μ_i, ν_i and k_4 in Eqs. (20) and (12), the exact travelling wave solutions for Eq. (4) are obtained in terms of hyperbolic, trigonometric and rational functions and discussed graphically in the following cases.

Case 1 For $l < 0$, the solutions involve hyperbolic functions as follows

Family 1

$$u(x, y, z, t) = \mu_0 + \frac{\left(2\,\rho\,k_1 l + 2\,k_1\rho\,m^2\right)}{\left(m - \sqrt{-l}\tanh\left(\sqrt{-l}\zeta\right)\right)}. \tag{28}$$

$$u(x, y, z, t) = \mu_0 + \frac{\left(2\,\rho\,k_1 l + 2\,k_1\rho\,m^2\right)}{\left(m - \sqrt{-l}\coth\left(\sqrt{-l}\zeta\right)\right)}, \tag{29}$$

where $\zeta = \frac{k_1x^\beta}{\Gamma(\beta+1)} + \frac{k_2y^\gamma}{\Gamma(\gamma+1)} + \frac{k_3z^\delta}{\Gamma(\delta+1)} - \frac{(4\,k_1{}^3k_3\rho^2 l - 3\,k_2)t^\alpha}{4\,k_1\Gamma(\alpha+1)}$. The solutions obtained in family 1 are shown graphically in Fig. 1. Figure 1a–f represents the three-dimensional (3D) and two-dimensional (2D) kink wave profiles for solution (28) and singular wave profiles for solution (29), respectively. The 3D plots help to compare the wave profiles for solutions in family 1 for fractional-order versus integral-order equation. The effect of variation of fractional parameters on the solutions obtained in the family 1 is depicted by the 2D plots. Note that there is no well-defined procedure to select a set of fractional-order parameters; hence, the solutions of fractional-order systems can be simulated by selecting various combinations of parameters.

Family 2

$$u(x,y,z,t) = \mu_0 + (2\,\rho\,k_1)\left(m - \sqrt{-l}\tanh\left(\sqrt{-l}\zeta\right)\right). \tag{30}$$

$$u(x,y,z,t) = \mu_0 + (2\,\rho\,k_1)\left(m - \sqrt{-l}\coth\left(\sqrt{-l}\zeta\right)\right). \tag{31}$$

The solutions obtained in the family 2 are shown graphically in Fig. 2. It describes the kink wave profiles for solution (30) in Fig. 2a–c and singular wave profiles for solution (31) in Fig. 2d–f. It reveals the effect of variation of fractional-order terms graphically on the solutions of family 2.

Case 2 For $l > 0$, solutions are obtained in trigonometric functions as follows.

Family 3

$$u(x,y,z,t) = \mu_0 + \frac{\left(2\,\rho\,k_1 l + 2\,k_1\rho\,m^2\right)}{\left(m + \sqrt{l}\tan\left(\sqrt{l}\zeta\right)\right)}. \tag{32}$$

$$u(x,y,z,t) = \mu_0 + \frac{\left(2\,\rho\,k_1 l + 2\,k_1\rho\,m^2\right)}{\left(m - \sqrt{l}\cot\left(\sqrt{l}\zeta\right)\right)}. \tag{33}$$

Figure 3 shows singular periodic wave profiles for solutions (32) and (33) obtained in family 3. Wave profiles of solutions are compared by 3D plots for integral values of fractional parameters. The effect of variation in fractional-order parameters on the wave profile of solutions is depicted by 2D plots in Fig. 3c, f.

Family 4

$$u(x,y,z,t) = \mu_0 + (2\,\rho\,k_1)\left(m + \sqrt{l}\tan\left(\sqrt{l}\zeta\right)\right). \tag{34}$$

$$u(x,y,z,t) = \mu_0 + (2\,\rho\,k_1)\left(m - \sqrt{l}\cot\left(\sqrt{l}\zeta\right)\right). \tag{35}$$

Figure 4 also shows singular periodic wave profiles for solutions (34) and (35) for the family 4 in terms of 3D and 2D representations.

Case 3 When $l = 0$, solutions appeared in the form of following rational functions.

Family 5

$$u(x,y,z,t) = \mu_0 + \frac{\left(2\,\rho\,k_1 l + 2\,k_1\rho\,m^2\right)\zeta}{(m\,\zeta - 1)}. \tag{36}$$

Family 6

$$u(x,y,z,t) = \mu_0 + (2\,\rho\,k_1)\left(m - \frac{1}{\zeta}\right). \tag{37}$$

The graphical analysis of solutions obtained for Eq. (4) in families 5 and 6 is shown in Fig. 5. It shows singular kink wave profiles for solutions (36) and (37). Figure 5a, d represents 3D wave profiles at $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$ and $\delta = 0.15$, and Fig. 5b, e depicts the 3D wave profiles at $\alpha = \beta = \gamma = \delta = 1$. The effect of fractional orders on the wave profile of the solutions is also shown by 2D representations in Fig. 5c, f.

The real-life application of fractional-order Eq. (4) helps to understand the rogue waves which were observed in deep sea, and this is a field of intensive research in oceanography [15,39].

Remark Solutions (28)–(37) for Eq. (4) are cross-checked and satisfied by using chain rule (13) in Maple software. It has been found that for parameters $\alpha = \beta = \gamma = \delta = 1$ and $\rho = k_1 = k_2 = k_3 = 1$, the solutions approach towards results as reported by [8,13,33,78,79] for integral-order YTSF equation. Hence, solutions presented in this paper are generalised as the fractional-order terms α, β, γ and δ can be set to any value lying between 0 and 1.

4 Linear dispersion analysis

In this section, the linear dispersion analysis of space–time fractional potential YTSF Eq. (4) is discussed using the definition of RL fractional derivative [49] with lower integral limit at $-\infty$ given by

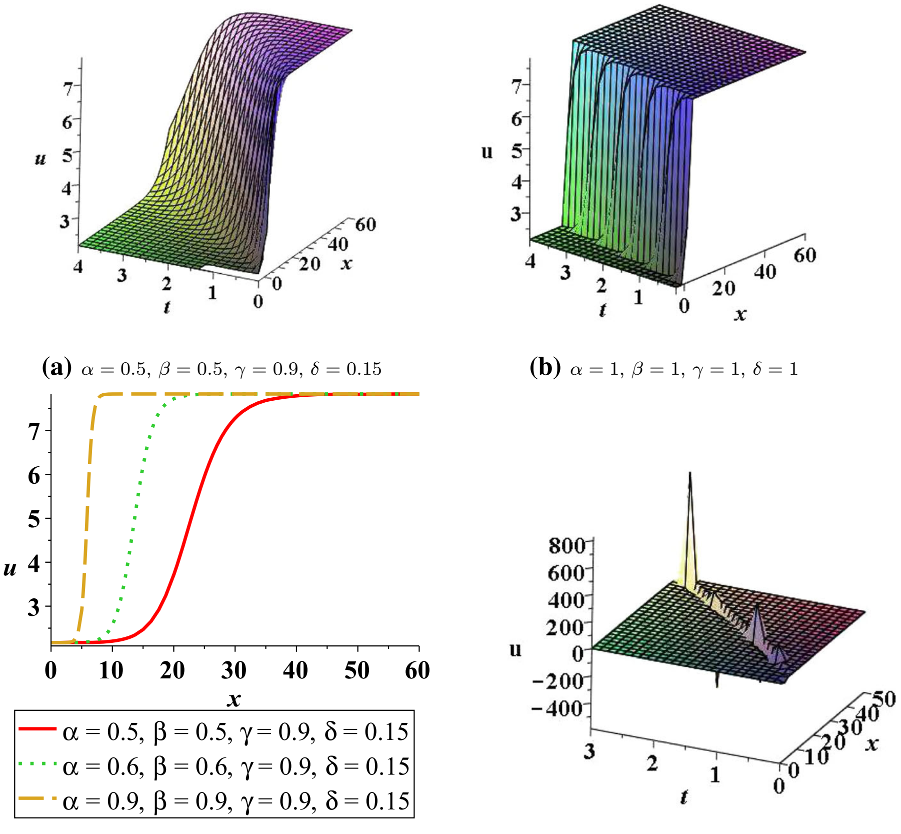
$$_{-\infty}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\infty}^{t} (t-\theta)^{n-\alpha-1} f(\theta)\,\mathrm{d}\theta,\, n-1<\alpha<n,\ n\in\mathbb{N}. \tag{38}$$

The RL fractional derivative (38) with lower terminal at $-\infty$ has the following property [49]

$$_{-\infty}D_t^{\alpha} e^{\lambda t} = \lambda^{\alpha} e^{\lambda t},\ \lambda > 0. \tag{39}$$

For linear analysis, the dispersive waves in the sinusoidal form are considered having periodic spatial and

Fig. 1 Wave profiles of family 1 representing **a**–**c** for solution (28) and **d**–**f** for solution (29) with $\mu_0 = 1$, $l = -2$, $k_1 = 1$, $k_2 = 1$, $k_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots

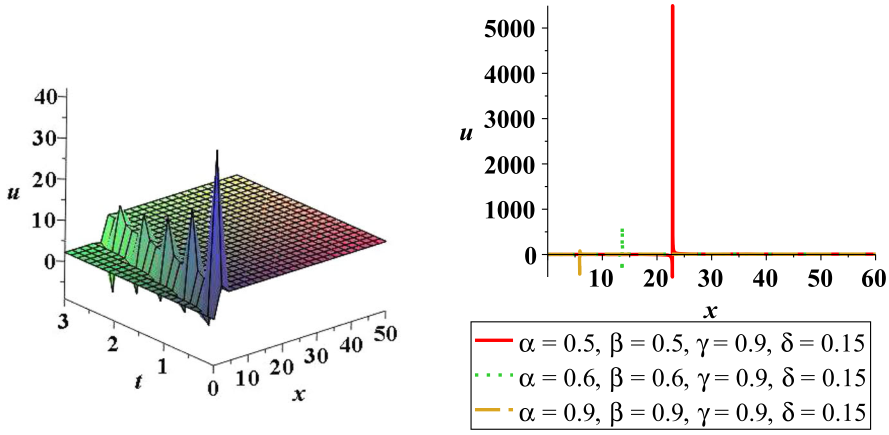


(a) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$

(b) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$

(c) Effect of fractional orders α, β, γ and δ

(d) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$



(e) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$

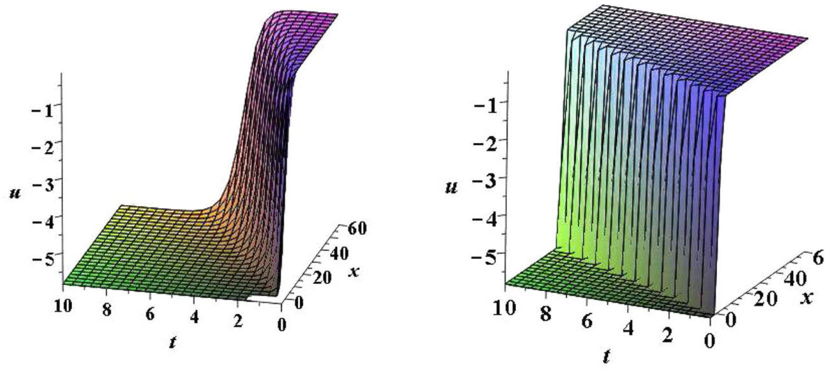
(f) Effect of fractional orders α, β, γ and δ

time dependence [2,16,23,36,42,43,66]. Let us consider $\psi(x, y, z, t)$ as the periodic wave function for a (3 + 1)-dimensional system, and it is given by

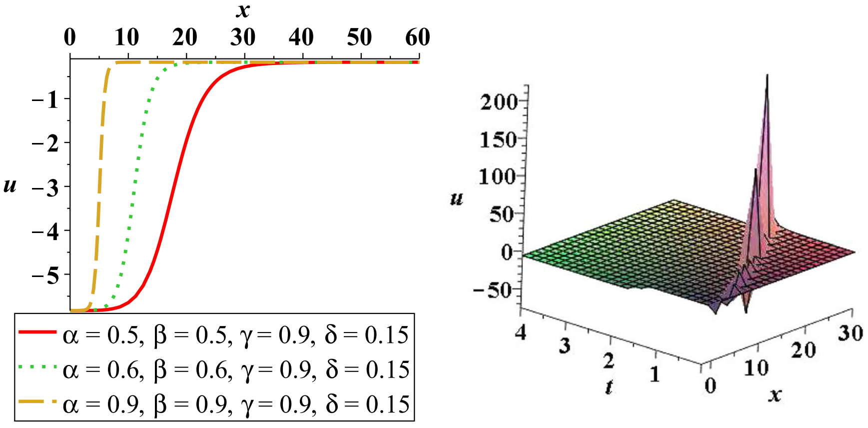
$$\psi(x, y, z, t) = \mathtt{Re}\{Ae^{i(\omega t - \mathbf{K}\cdot\mathbf{r})}\}, \tag{40}$$

where Re denotes the real part of the wave function, $i = \sqrt{-1}$, A is the complex amplitude, $\mathbf{K} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$ is the wave vector, and $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ is the radial vector. Here, subscripts x, y, z describe the component of a 3D vector along three orthogonal directions and $\hat{i}$,

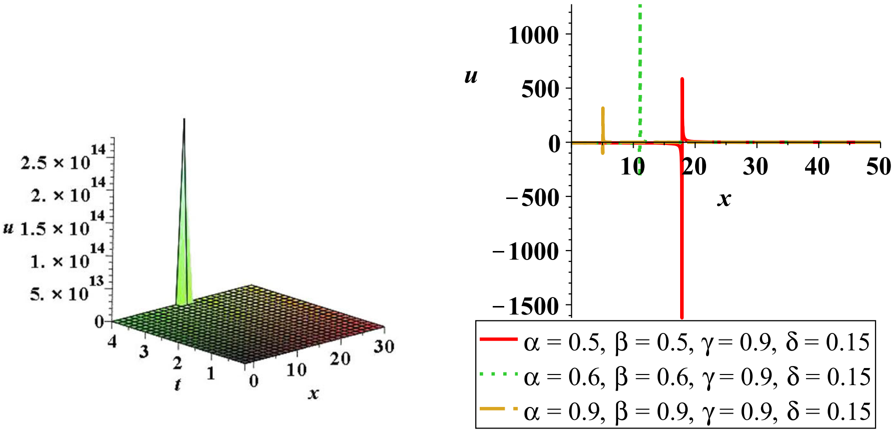
Fig. 2 Wave profiles of family 2 representing **a**–**c** for solution (30) and **d**–**f** for solution (31) with $\mu_0 = 1$, $l = -2$, $k_1 = 1$, $k_2 = 1$, $k_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots



(a) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$ **(b)** $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$

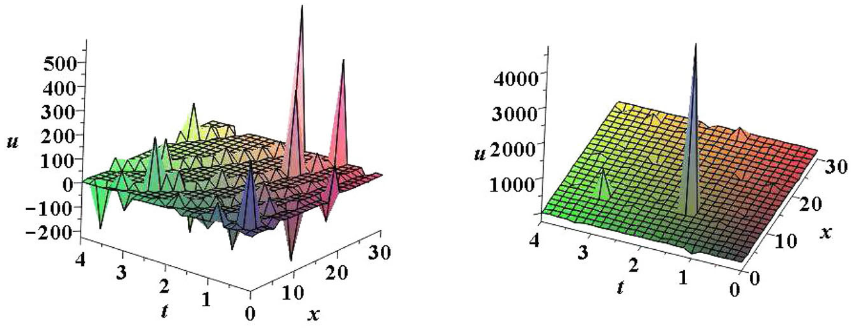


(c) Effect of fractional orders α, β, γ and δ **(d)** $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$

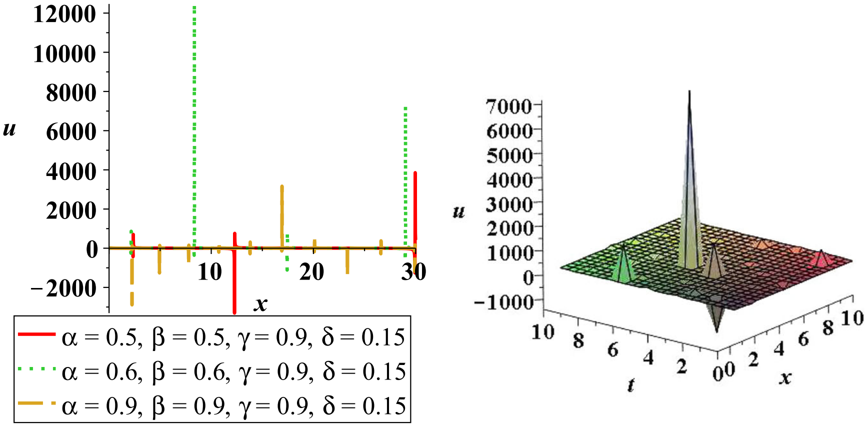


(e) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$ **(f)** Effect of fractional orders α, β, γ and δ

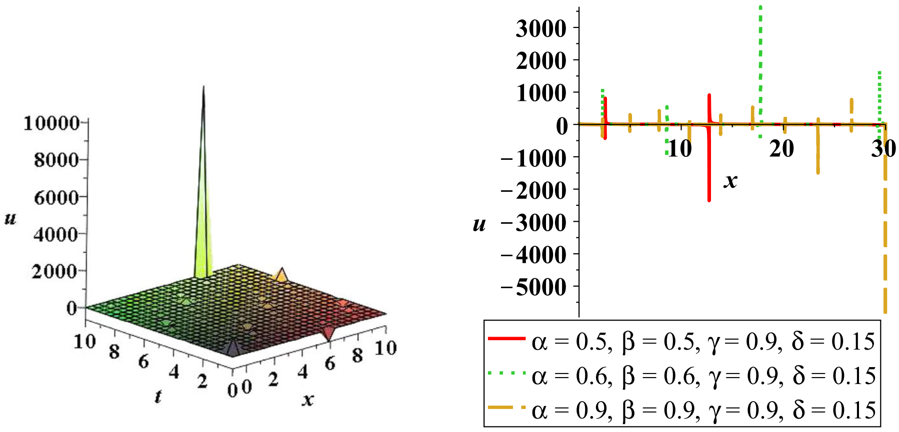
Fig. 3 Wave profiles of family 3 representing **a**–**c** for solution (32) and **d**–**f** for solution (33) with $\mu_0 = 1$, $l = 2$, $k_1 = 1$, $k_2 = 1$, $k_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots



(a) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$ **(b)** $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$

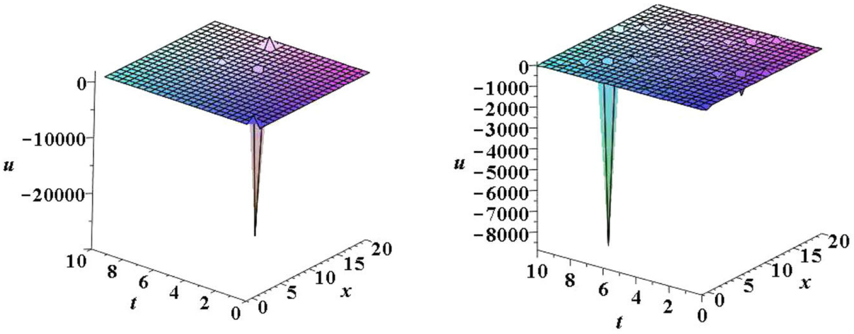


(c) Effect of fractional orders α, β, γ and δ **(d)** $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$

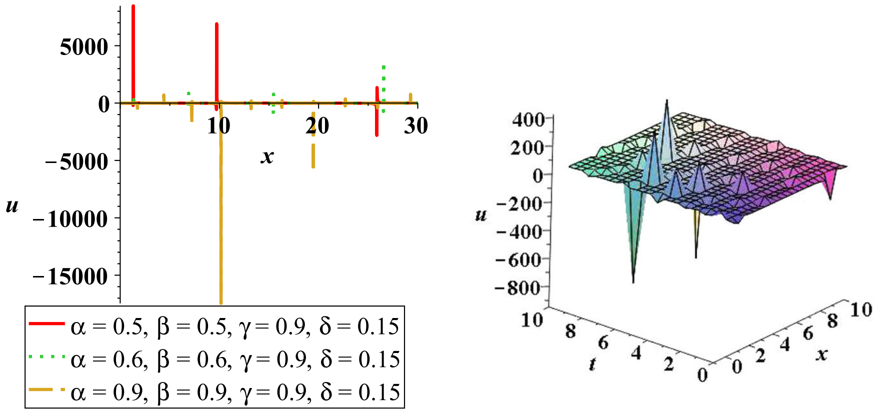


(e) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$ **(f)** Effect of fractional orders α, β, γ and δ

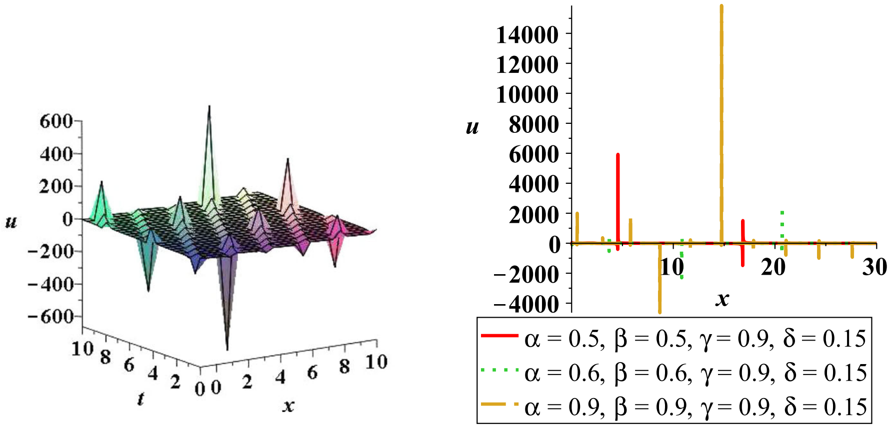
Fig. 4 Wave profiles of family 4 representing **a**–**c** for solution (34) and **d**–**f** for solution (35) with $\mu_0 = 1$, $l = 2$, $k_1 = 1$, $k_2 = 1$, $k_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots



(a) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$ **(b)** $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$



(c) Effect of fractional orders α, β, γ and δ **(d)** $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$



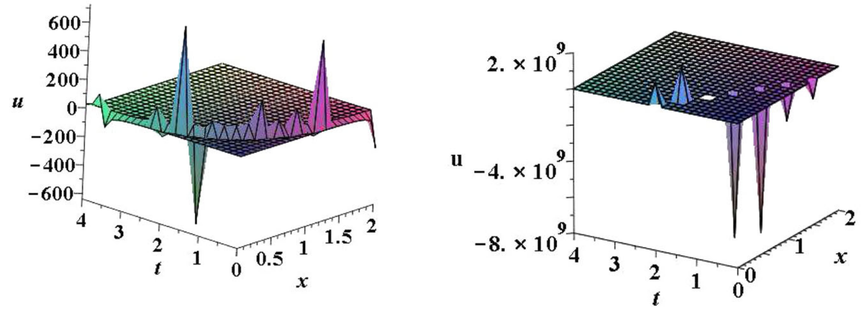
(e) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$ **(f)** Effect of fractional orders α, β, γ and δ

$\hat{j}$, $\hat{k}$ are unit vectors along x, y, z directions, respectively. The $\cdot$ between $\mathbf{K}$ and $\mathbf{r}$ stands for the scalar product of two vectors. The relation between ω and vector K is called the dispersion relation and is represented as follows

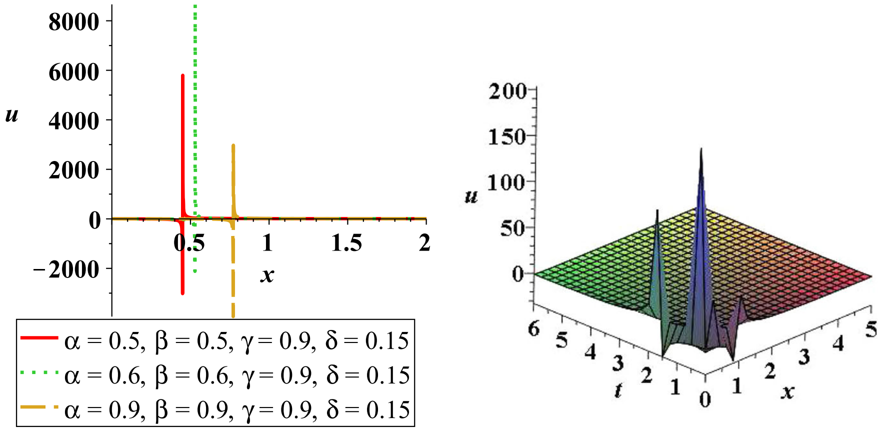
$$\mathcal{D}(\omega, \mathbf{K}) = 0, \tag{41}$$

where $\mathcal{D}$ is the real function of ω and $\mathbf{K}$. Such a relation, in general, is satisfied by certain ω, $\mathbf{K} \in \mathbb{C}$. This equation can be solved explicitly in terms of a real parameter ω by means of the following condition

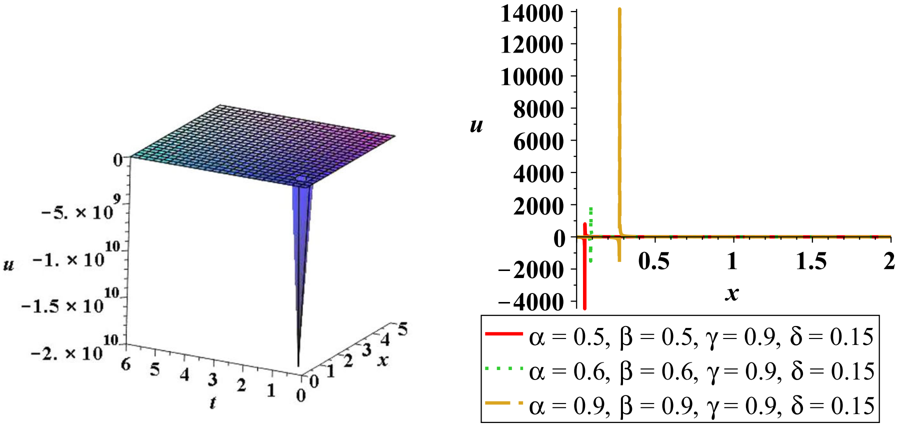
Fig. 5 Wave profiles of families 5 and 6 representing **a**–**c** for solution (36) and **d**–**f** for solution (37) with $\mu_0 = 1$, $l = 0$, $k_1 = 1$, $k_2 = 1$, $k_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots



(a) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$ **(b)** $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$



(c) Effect of fractional orders α, β, γ and δ **(d)** $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$



(e) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$ **(f)** Effect of fractional orders α, β, γ and δ

$$\bar{\omega}(\mathbf{K}) = \bar{\omega}(k_x, k_y, k_z) \in \mathbb{C}, \;\; k_x, k_y, k_z \in \mathbb{R}. \tag{42}$$

Then, the solution of the dynamical equation can be written as follows

$$\psi(x, y, z, t; K) = \mathtt{Re}\{A(\mathbf{K}) e^{i(\bar{\omega}t - \mathbf{K}\cdot\mathbf{r})}\}. \tag{43}$$

The phase and group velocities from dispersion are defined as follows

$$\begin{aligned} v_p(\mathbf{K}) =& \frac{\mathbf{K}\,\mathtt{Re}\bar{\omega}(\mathbf{K})}{|\mathbf{K}|^2}, \\ v_g(\mathbf{K}) =& \nabla_{\mathbf{K}}\,\mathtt{Re}\bar{\omega}(\mathbf{K}), \end{aligned} \tag{44}$$

where $\nabla_{\mathbf{K}}$ denotes gradient and $|\mathbf{K}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

Equation (43) shows that $\psi(x, y, z, t; \mathbf{K})$ is oscillating sinusoidally in space with a wavelength $\lambda = \frac{2\pi}{|\mathbf{K}|}$, and with time, the sinusoidal variation depends on whether the frequency ω is real or imaginary. Thus,

$$\bar{\omega}(\mathbf{K}) = \omega_r(\mathbf{K}) + i\,\omega_i(\mathbf{K}), \tag{45}$$

where $\omega_r(\mathbf{K}) = \mathtt{Re}\,(\bar{\omega})$ and $\omega_i(\mathbf{K}) = \mathtt{Im}\,(\bar{\omega})$. If $\omega_i \geq 0$, $\gamma(K) = \omega_i$ is the time-damping factor. Now, consider linearised form of Eq. (4) as given by

$$-4\,\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right) + \frac{\partial^{3\beta}}{\partial x^{3\beta}}\left(\frac{\partial^{\delta} u}{\partial z^{\delta}}\right) + 3\,\frac{\partial^{2\gamma} u}{\partial y^{2\gamma}} = 0. \tag{46}$$

The sinusoidal solution for the linear system of Eq. (46) is given by

$$u(x, y, z, t) = A\, e^{i(\bar{\omega}t - \mathbf{K}\cdot\mathbf{r})}, \tag{47}$$

where A is the amplitude of u. Substituting Eq. (47) into Eq. (46) and using Fourier transform for fractional-order derivative terms [49] with RL fractional derivative (38) gives

$$\begin{aligned} &\left(-4(i\bar{\omega})^{\alpha}(-ik_x)^{\beta} + (-ik_x)^{3\beta}(-ik_z)^{\delta} + 3(-ik_y)^{2\gamma}\right) \\ &\quad A\, e^{i(\bar{\omega}t - K\cdot r)} = 0. \end{aligned} \tag{48}$$

Solving Eq. (48) for $\bar{\omega}$, we get

$$\bar{\omega} = (\omega_0)^{1/\alpha}, \tag{49}$$

where

$$\begin{aligned} \omega_0 = \frac{1}{4}\Big(&(-1)^{2\beta+\delta}\, {k_x}^{2\beta} {k_z}^{\delta}\, (\cos(\theta_1) + i \sin(\theta_1)) \\ &+ 3\, (-1)^{2\gamma-\beta}\, {k_y}^{2\gamma} {k_x}^{-\beta}\, (\cos(\theta_2) + i\sin(\theta_2))\Big), \end{aligned}$$

$$\theta_1 = 1/2\; (2\beta + \delta - \alpha)\,\pi, \theta_2 = 1/2\; (2\gamma - \beta - \alpha)\,\pi.$$

The expressions for phase and group velocities using dispersion relation are given by

$$\begin{aligned} v_p(\mathbf{K}) &= \frac{k_x\,\hat{i} + k_y\,\hat{j} + k_z\,\hat{k}}{k_x^2 + k_y^2 + k_z^2}\,(\omega_0)^{1/\alpha}, \\ v_g(\mathbf{K}) &= \frac{\partial}{\partial k_x}{\omega_0}^{1/\alpha}\,\hat{i} + \frac{\partial}{\partial k_y}{\omega_0}^{1/\alpha}\,\hat{j} + \frac{\partial}{\partial k_z}{\omega_0}^{1/\alpha}\,\hat{k}. \end{aligned} \tag{50}$$

Thus, magnitude for phase and group velocity components are given as follows

$$\begin{aligned} |v_p| &= \frac{(4)^{-1/\alpha}\,{\omega_0}^{1/\alpha}}{\sqrt{{k_x}^2 + {k_y}^2 + {k_z}^2}}, \\ |v_g| &= \frac{(4)^{-1/\alpha}\,{\omega_0}^{(1/\alpha)-1}}{\alpha} \\ &\quad \left({\omega_1}^2 + 36\,\frac{(-1)^{4\gamma-2\beta}\,{k_y}^{4\gamma}\gamma^2{k_x}^{-2\beta}\,(\cos(2\theta_2) + i\sin(2\theta_2))}{{k_y}^2}\right. \\ &\quad \left. + \frac{(-1)^{4\beta+2\delta}\,{k_x}^{4\beta}{k_z}^{2\delta}\delta^2\,(\cos(2\theta_1) + i\sin(2\theta_1))}{{k_z}^2}\right)^{1/2}, \end{aligned} \tag{51}$$

where

$$\begin{aligned} \omega_1 = \frac{1}{k_x}\Big(&2\,(-1)^{2\beta+\delta}\,{k_x}^{2\beta}\beta\,{k_z}^{\delta}\,(\cos(\theta_1) + i\sin(\theta_1)) \\ &- 3\,(-1)^{2\gamma-\beta}\,{k_y}^{2\gamma}{k_x}^{-\beta}\beta\;(\cos(\theta_2) + i\sin(\theta_2))\Big). \end{aligned} \tag{52}$$

Equation (51) shows that $|v_p|$ is not constant; this shows that medium is dispersive. The group velocity and phase velocity magnitudes are plotted in Fig. 6 for different values of α, β, γ and δ. It has been noticed that the imaginary part of phase velocity is greater than the imaginary part of group velocity for $\alpha = 0.75$, $\beta = 0.4$, $\gamma = 0.9$ and $\delta = 0.6$ values. It implies that the sinusoidal waves with large wavelength travel faster than the waves with smaller wavelength. Thus, it suggests the normal dispersion of waves. In contrast, the real part of group velocity is greater than real part of phase velocity for the same set of parameters; thus, it indicates the anomalous dispersion of waves. The relation given in Eq. (51) is not linear in k_x, k_y and k_z; thus, it suggests that wave packet gets distorted in shape.

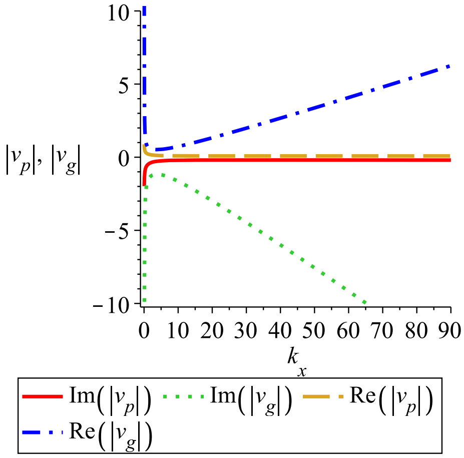


Fig. 6 Comparison between phase and group velocities with $\alpha = 0.75$, $\beta = 0.4$, $\gamma = 0.9$, $\delta = 0.6$, $k_y = 1$, $k_z = 1$

5 Conclusion

The paper successfully applied the improved F-expansion method for finding exact solutions of (3 + 1)-dimensional space–time fractional potential YTSF equation. The solutions are obtained in terms of hyperbolic, trigonometric and rational functions, and further, the effect of fractional-order parameters on the wave profile of solutions is described in terms of 2D and 3D plots. The solutions possess kink, periodic and singular wave profiles. The linear analysis describes the dependence of anomalous and normal dispersion of waves on different values of fractional orders.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Formulation of space–time fractional potential YTSF Eq. (4)

The Lagrangian [3,24] for the potential YTSF Eq. (3) can be obtained as follows

$$\mathcal{L}(u_x, u_y, u_z, u_t, u_{xx}, u_{xz}) = 2u_x u_t + \frac{1}{2}u_{xx}u_{xz} - \frac{3}{2}u_y^2 - u_x^2 u_z. \tag{53}$$

Similarly, the Lagrangian for space–time fractional Eq. (4) is obtained as follows

$$F(D_x^\beta u, D_y^\gamma u, D_z^\delta u, D_t^\alpha u, D_x^{2\beta} u, D_x^\beta D_z^\delta u) = 2D_x^\beta u D_t^\alpha u + \frac{1}{2}D_x^{2\beta} u D_x^\beta D_z^\delta u - \frac{3}{2}(D_y^\gamma u)^2 - (D_x^\beta u)^2 D_z^\delta u, \tag{54}$$

where D_x^β, D_y^γ, D_z^δ, D_t^α, $D_x^{2\beta}$ denote the fractional derivatives $\frac{\partial^\beta}{\partial x^\beta}$, $\frac{\partial^\gamma}{\partial y^\gamma}$, $\frac{\partial^\delta}{\partial z^\delta}$, $\frac{\partial^\alpha}{\partial t^\alpha}$, $\frac{\partial^{2\beta}}{\partial x^{2\beta}}$, respectively. The functional for Eq. (4) can be written as follows

$$J_F(u) = \int_R (\mathrm{d}x)^\beta \int_R (\mathrm{d}y)^\gamma \int_R (\mathrm{d}z)^\delta \int_R (\mathrm{d}t)^\alpha \cdot F(D_x^\beta u, D_y^\gamma u, D_z^\delta u, D_t^\alpha u, D_x^{2\beta} u, D_x^\beta Dz^\delta u) \tag{55}$$

The variation of functional for Eq. (55) is given by using the method given in [4,5,24] and is obtained as follows

$$\Delta J_F(u) = \int_R (\mathrm{d}x)^\beta \int_R (\mathrm{d}y)^\gamma \int_R (\mathrm{d}z)^\delta \int_R (\mathrm{d}t)^\alpha \cdot \left[\frac{\partial F}{\partial D_t^\alpha u}\Delta D_t^\alpha u + \frac{\partial F}{\partial D_x^\beta u}\Delta D_x^\beta u + \frac{\partial F}{\partial D_y^\gamma u}\Delta D_y^\gamma u + \frac{\partial F}{\partial D_z^\delta u}\Delta D_z^\delta u + \frac{\partial F}{\partial D_x^{2\beta} u}\Delta D_x^{2\beta} u + \frac{\partial F}{\partial D_x^\beta D_z^\delta u}\Delta D_x^\beta D_z^\delta u\right], \tag{56}$$

where $\int_a^\tau (\mathrm{d}\tau)^j f(\tau) = j\int_a^t \mathrm{d}(\tau)(t-\tau)^j f(\tau)$. Using the fractional integration by parts

$$\int_a^b (\mathrm{d}Z)^j f(Z) D_Z^j g(Z) = \Gamma(1+j)\left[g(Z)f(Z)|_a^b - \int_a^b \mathrm{d}(Z)^j g(Z) D_Z^j f(Z)\right], \quad f(Z), g(Z) \in [a, b], \tag{57}$$

we can obtain

$$\Delta J_F(u) = \int_R (\mathrm{d}x)^\beta \int_R (\mathrm{d}y)^\gamma \int_R (\mathrm{d}z)^\delta \int_R (\mathrm{d}t)^\alpha \cdot \left[-D_t^\alpha\left(\frac{\partial F}{\partial D_t^\alpha u}\right) - D_x^\beta\left(\frac{\partial F}{\partial D_x^\beta u}\right) - D_y^\gamma\left(\frac{\partial F}{\partial D_y^\gamma u}\right) - D_z^\delta\left(\frac{\partial F}{\partial D_z^\delta u}\right)\right.$$

$$+D_x^{2\beta}\left(\frac{\partial F}{\partial D_x^{2\beta}u}\right)+D_x^{\beta}D_z^{\delta}\left(\frac{\partial F}{\partial D_x^{\beta}D_z^{\delta}u}\right)\Bigg]. \tag{58}$$

Optimising the variation Eq. (56), $\Delta J_F(u)=0$, we can obtain the Euler–Lagrangian equation for Eq. (4) as follows

$$\begin{aligned}&-D_t^{\alpha}\left(\frac{\partial F}{\partial D_t^{\alpha}u}\right)-D_x^{\beta}\left(\frac{\partial F}{\partial D_x^{\beta}u}\right)-D_y^{\gamma}\left(\frac{\partial F}{\partial D_y^{\gamma}u}\right)\\&-D_z^{\delta}\left(\frac{\partial F}{\partial D_z^{\delta}u}\right)+D_x^{2\beta}\left(\frac{\partial F}{\partial D_x^{2\beta}u}\right)\\&+D_x^{\beta}D_z^{\delta}\left(\frac{\partial F}{\partial D_x^{\beta}D_z^{\delta}u}\right)=0.\end{aligned} \tag{59}$$

Substituting Eq. (54) into Eq. (59), we obtain space–time fractional potential YTSF Eq. (4) as follows

$$\begin{aligned}&-4\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right)+\frac{\partial^{3\beta}}{\partial x^{3\beta}}\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right)+3\frac{\partial^{2\gamma}u}{\partial y^{2\gamma}}\\&+4\left(\frac{\partial^{\beta}u}{\partial x^{\beta}}\right)\left(\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right)\right)\\&+2\left(\frac{\partial^{2\beta}u}{\partial x^{2\beta}}\right)\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right)=0.\end{aligned} \tag{60}$$

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