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On explicit exact solutions of variable-coefficient time-fractional generalized fifth-order Korteweg-de Vries equation

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Abstract. We investigate the variable-coefficient time-fractional generalized fifth-order Korteweg-de Vries equation for admissible forms of the variable coefficients under the condition of invariance, and derive certain explicit exact solutions for the reduced ordinary differential equations of fractional order.

1 Introduction

Fractional calculus has extensible applications in various fields such as viscoelasticity and damping, heat transfer, signal processing, electrical engineering, traffic systems, control system, economy and finance [1–5]. Therefore, in many cases, the real physical phenomenon could be modeled by applying the theory of fractional-order derivatives and integrals [6–8]. The symmetry method based on finding symmetries and reductions of differential equations lead to their exact solutions. Initially, the symmetry method [9,10] was applied to integer-order PDE, but in the last few years this technique has been applied to the analysis of fractional-order partial differential equations [11–17]. Apart from the symmetry method, few numerical and analytic techniques are also availiable [18–21].

In various physical situations when the inhomogeneities of media are considered, nonlinear variable-coefficient fractional PDEs provide more realistic models than their constant-coefficient counterparts. So it is of great importance to find explicit exact solutions of fractional nonlinear PDEs with variable coefficients. Only a few time-fractional variable-coefficient FDEs [22] have been studied using the Lie symmetry method. But explicit exact solutions of higher-order time-fractional variable-coefficient PDEs has not been discussed till now. In [23], the Lie group method has been applied to constant-coefficients time-fractional generalized fifth-order Korteweg-de Vries equation.

In the present work, we derive explicit exact solutions of the variable-coefficient time-fractional generalized fifth-order Korteweg-de Vries equation

$$\partial_t^\alpha u + \delta(t)u^2 u_x + \gamma(t)u_x u_{xx} + \beta(t)uu_{xxx} + u_{xxxxx} = 0. \tag{1}$$

In eq. (1), u is a function of x and t, $0 < \alpha \leqslant 1$, and $\beta(t)$, $\gamma(t)$, $\delta(t)$ are non zero functions of t, and $\partial_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha}$ denotes the α-th–order Riemann-Liouville fractional derivatives of u. The generalized fifth-order KDV equation can describe the interaction between a water wave and a floating ice cover in the river channels [24], shallow-water waves, internal gravity waves, and so on [25–28].

The work in this paper is organized as follows: sect. 2 is devoted to the preliminaries and symmetry approach. In sect. 3, we transform the fractional partial differential eq. (1) into fractional ODE, and find some explicit exact solutions. Section 4 provides the concluding remarks.

2 Preliminaries and symmetries analysis

In the literature, many definitions of fractional derivative, like Riemann-Liouville fractional derivative, Caputo fractional derivative, Miller-Ross fractional derivative, [6,7] exist. Here we consider the Riemann-Liouville fractional

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derivative of order $\alpha > 0$, and it is defined by [15,29]:

$$\mathrm{D}^\alpha g(t) = \begin{cases} \dfrac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}, & \alpha = k, \\ \dfrac{\mathrm{d}^k}{\mathrm{d}t^k}\mathrm{I}^{\alpha-k} g(t), & 0 \le k-1 < \alpha < k, \end{cases} \tag{2}$$

where $k \in \mathbb{N}$, $\mathrm{I}^\beta g(t)$ is the Riemann-Liouville fractional integral [29] of order β defined as follows:

$$\mathrm{I}^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\lambda)^{\beta-1} g(\lambda)\mathrm{d}\lambda, \quad \beta > 0. \tag{3}$$

Similarly, the Riemann-Liouville partial fractional derivative of the function $u(x,t)$ of order α with respect to variable t can be defined as follows [12,17]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^k u}{\partial t^k}, & \alpha = k, \\ \dfrac{\partial^k}{\partial t^k} \dfrac{1}{\Gamma(k-\alpha)} \displaystyle\int_0^t (t-\lambda)^{k-\alpha-1} u(x,\lambda)\mathrm{d}\lambda, & 0 \le k-1 < \alpha < k. \end{cases} \tag{4}$$

Now we introduce some formulas needed to find the Lie point symmetry analysis of FPDEs [14–16]. The general form of time fractional PDE with dependent variable u, and independent variables (x,t) is as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right), \quad \alpha > 0. \tag{5}$$

Assume the invariance of eq. (7) under the one-parameter (ϵ) Lie group of transformations [9,10] given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\zeta(x,t,u) + O(\epsilon^2), \\ \frac{\partial^\alpha u^*}{\partial t^{*\alpha}} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\zeta^{\alpha,t} + O(\epsilon^2), \\ \frac{\partial u^*}{\partial x^*} &= \frac{\partial u}{\partial x} + \epsilon\zeta^x + O(\epsilon^2), \end{aligned} \tag{6}$$

where (ξ, τ, ζ) is set of infinitesimals, which are functions of (x,t,u), $\zeta^{\alpha,t}$ is the extended infinitesimal of order α, and ζ^x is the extended infinitesimal of order 1, and so on. The corresponding prolongation of infinitesimal generator,

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \zeta(x,t,u)\frac{\partial}{\partial u}, \tag{7}$$

can be defined as follows:

$$\tilde{V} = V + \zeta^{\alpha,t}\frac{\partial}{\partial u_t^\alpha} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^{xx}\frac{\partial}{\partial u_{xx}} + \dots. \tag{8}$$

The extended infinitesimal $\zeta^{\alpha,t}$ related to Riemann-Liouville fractional derivative is given by [12,16–18]

$$\zeta^{\alpha,t} = \mathrm{D}_t^\alpha(\zeta) + \xi\mathrm{D}_t^\alpha(u_x) - \mathrm{D}_t^\alpha(\xi u_x) + \mathrm{D}_t^\alpha(\mathrm{D}_t(\tau)u) - \mathrm{D}_t^{\alpha+1}(\tau u) + \tau\mathrm{D}_t^{\alpha+1}(u). \tag{9}$$

In view of the generalized Leibnitz rule [29] of the form

$$\mathrm{D}_t^\alpha\left(g(t)h(t)\right) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \mathrm{D}_t^k g(t) \mathrm{D}_t^{\alpha-k} h(t), \quad \alpha > 0, \tag{10}$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)},$$

we arrive at

$$\begin{aligned}\mathrm{D}_t^{\alpha}\left(\xi u_x\right) &= \sum_{k=0}^{\infty}\binom{\alpha}{k}\mathrm{D}_t^{k}(\xi)\mathrm{D}_t^{\alpha-k}(u_x)\\ &= \xi\mathrm{D}_t^{\alpha}(u_x)+\sum_{k=1}^{\infty}\binom{\alpha}{k}\mathrm{D}_t^{k}(\xi)\mathrm{D}_t^{\alpha-k}(u_x),\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mathrm{D}_t^{\alpha}(\mathrm{D}_t(\tau)u)-\mathrm{D}_t^{\alpha+1}(\tau u)+\tau\mathrm{D}_t^{\alpha+1}(u) &= \tau\mathrm{D}_t^{\alpha}(u_t)-\mathrm{D}_t^{\alpha}(\tau u_t)\\ &= -\alpha\mathrm{D}_t(\tau)\mathrm{D}_t^{\alpha}(u)-\sum_{k=1}^{\infty}\binom{\alpha}{k+1}\mathrm{D}_t^{k+1}(\tau)\mathrm{D}_t^{\alpha-k}(u).\end{aligned} \tag{12}$$

On substituting eqs. (11) and (12) in expression (9), the following relation is obtained:

$$\zeta^{\alpha,t}=\mathrm{D}_t^{\alpha}(\zeta)-\alpha\mathrm{D}_t(\tau)\mathrm{D}_t^{\alpha}(u)-\sum_{k=1}^{\infty}\binom{\alpha}{k}\mathrm{D}_t^{k}(\xi)\mathrm{D}_t^{\alpha-k}(u_x)-\sum_{k=1}^{\infty}\binom{\alpha}{k+1}\mathrm{D}_t^{k+1}(\tau)\mathrm{D}_t^{\alpha-k}(u). \tag{13}$$

By using the generalized Leibniz rule and generalized chain rule [7], the term $\mathrm{D}_t^{\alpha}(\zeta)$ in (13) can be written as follows:

$$\mathrm{D}_t^{\alpha}(\zeta)=\frac{\partial^{\alpha}\zeta}{\partial t^{\alpha}}+\left(\zeta_u\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-u\frac{\partial^{\alpha}\zeta_u}{\partial t^{\alpha}}\right)+\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{\partial^{k}\zeta_u}{\partial t^{k}}\mathrm{D}_t^{\alpha-k}(u)+\kappa, \tag{14}$$

where

$$\kappa=\sum_{k=2}^{\infty}\sum_{m=2}^{k}\sum_{n=2}^{m}\sum_{r=0}^{n-1}\binom{\alpha}{k}\binom{k}{m}\binom{n}{r}\frac{1}{n!}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u)^{n-r}\frac{\partial^{k-m+n}\zeta}{\partial t^{k-m}\partial u^n}.$$

Therefore, the explicit form of the extended infinitesimal $\zeta^{\alpha,t}$ for FPDE of the form (5) can be obtained as follows:

$$\zeta^{\alpha,t}=\frac{\partial^{\alpha}\zeta}{\partial t^{\alpha}}+(\zeta_u-\alpha\mathrm{D}_t(\tau))\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-u\frac{\partial^{\alpha}\zeta_u}{\partial t^{\alpha}}+\kappa-\sum_{k=1}^{\infty}\binom{\alpha}{k}\mathrm{D}_t^{k}(\xi)\mathrm{D}_t^{\alpha-k}u_x+\sum_{k=1}^{\infty}\left[\binom{\alpha}{k}\frac{\partial^{k}\zeta_u}{\partial t^{k}}-\binom{\alpha}{k+1}\mathrm{D}_t^{k+1}\tau\right]\mathrm{D}_t^{\alpha-k}(u). \tag{15}$$

Since in Riemann-Liouville fractional-derivative operator (4), the lower limit of the integral is fixed, it should be invariant under (6). Such invariance condition yields

$$\tau(x,t,u)|_{t=0}=0. \tag{16}$$

The invariance criterion of the time-fractional generalized fifth-order KDV equation (1) can be written as follows:

$$\tilde{V}(\Delta)|_{\Delta=0}=0, \tag{17}$$

where Δ: $\partial_t^{\alpha}u+\beta(t)uu_{xxx}+\gamma(t)u_xu_{xx}+\delta(t)u^2u_x+u_{xxxxx}=0$.

On substituting Δ in invariance condition (17), we have

$$\begin{aligned}&[\zeta^{\alpha,t}+\beta(t)\left(\zeta u_{xxx}+u\zeta^{xxx}\right)+\gamma(t)\left(\zeta^{x}u_{xx}+u_x\zeta^{xx}\right)+\delta(t)\left(2u\zeta u_x+u^2\zeta^{x}\right)\\ &+\tau\left(\beta'(t)uu_{xxx}+\gamma'(t)u_xu_{xx}+\delta'(t)u^2u_x\right)+\zeta^{xxxxx}]|_{(1)}=0.\end{aligned} \tag{18}$$

Using the extended infinitesimals, and equating various powers of derivatives of u to zero, we obtain the following set of determining equations:

$$\begin{aligned}&\xi_t=\xi_u=0,\qquad \tau_x=\tau_u=0,\qquad \zeta_{uu}=0,\qquad \alpha\tau_t-5\xi_x=0,\\ &\binom{\alpha}{k}\partial_t^{k}\zeta_u-\binom{\alpha}{k+1}\mathrm{D}_t^{k+1}\tau=0,\quad k\in\mathbb{N},\\ &\partial_t^{\alpha}\zeta-u\partial_t^{\alpha}\zeta_u+\zeta_{5x}+\beta(t)u\zeta_{xxx}+\gamma(t)u^2\zeta_x=0,\\ &\beta(t)u\left(\alpha\tau_t-3\xi_x\right)+\tau\beta'(t)u+\beta(t)\zeta+10\left(\zeta_{xxu}-\xi_{xxx}\right)=0,\\ &\gamma(t)\left(\alpha\tau_t-3\xi_x+\zeta_u\right)+\tau\gamma'(t)=0,\\ &\delta(t)u^2\left(\alpha\tau_t-3\xi_x\right)+\beta(t)u\left(\zeta_{xxu}-\xi_{xxx}\right)+\gamma(t)\zeta_{xx}+\tau u^2\delta'(t)+2u\zeta\delta(t)=0.\end{aligned} \tag{19}$$

On solving the above determining equations, we obtain the infinitesimals as follows:

$$\zeta = d_1 u, \qquad \xi = \frac{d_2 x}{5} + d_3, \qquad \tau = \frac{d_2 t}{\alpha}, \tag{20}$$

where d_1, d_2, d_3 are arbitrary constants and $\beta(t)$, $\gamma(t)$, $\delta(t)$ are obtained from the following equations:

$$\begin{aligned} \beta(t)\left(\frac{2}{5}d_1 + d_2\right) + \left(\frac{d_1 t}{\alpha}\right)\beta'(t) = 0,\\ \gamma(t)\left(\frac{2}{5}d_1 + d_2\right) + \left(\frac{d_1 t}{\alpha}\right)\gamma'(t) = 0,\\ \delta(t)\left(\frac{4}{5}d_1 + 2d_2\right) + \left(\frac{d_1 t}{\alpha}\right)\delta'(t) = 0. \end{aligned} \tag{21}$$

The corresponding symmetry generators are as follows:

$$V_1 = u\partial_u, \qquad V_2 = \frac{x}{5}\partial_x + \frac{t}{\alpha}\partial_t, \qquad V_3 = \partial_x. \tag{22}$$

3 Symmetry reduction and some explicit exact solutions

The left-hand sided Erdélyi-Kober fractional differential operator [30,31] is defined as follows:

$$\left(\mathcal{P}_\delta^{\varrho,\alpha} g\right)(\theta) := \prod_{i=0}^{k-1}\left(\varrho + i - \frac{1}{\delta}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\left(\mathcal{K}_\delta^{\varrho+\alpha,k-\alpha} g\right)(\theta), \tag{23}$$

for $\theta > 0$, $\delta > 0$, $\alpha > 0$, and

$$k = \begin{cases} [\alpha] + 1, & \text{if } \alpha \notin \mathbb{N},\\ \alpha, & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

where

$$\left(\mathcal{K}_\delta^{\varrho,\alpha} g\right)(\theta) := \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^\infty (\lambda - 1)^{\alpha-1}\lambda^{-(\varrho+\alpha)} g(\theta\lambda^{\frac{1}{\delta}})\mathrm{d}\lambda, & \text{if } \alpha > 0,\\ g(\theta), & \text{if } \alpha = 0, \end{cases} \tag{24}$$

is the extended left-hand-sided Erdélyi-Kober fractional integral operator.

Before finding explicit solutions of eq. (1), we use the concept of optimal system [9] for obtaining inequivalent symmetry reductions. The optimal system for eq. (1) has the following components:

$$\Delta_1 = V_1 + \mu V_2, \qquad \Delta_2 = V_2, \qquad \Delta_3 = V_3.$$

Case i) Generator Δ_1

For simplicity, let $\mu = 1$ and solving the characteristics equations

$$\frac{\mathrm{d}x}{x/5} = \frac{\mathrm{d}t}{t/\alpha} = \frac{\mathrm{d}u}{u},$$

we have the following similarity variable and similarity transformation:

$$\theta = xt^{-\frac{\alpha}{5}}, \qquad u = x^5 G(\theta). \tag{25}$$

The Riemann-Liouville fractional derivative of $u = x^5 G(\theta)$ where $\theta = xt^{-\frac{\alpha}{5}}$ with respect to t for $k-1 < \alpha < k$ ($k \in \mathbb{N}$) can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^k}{\partial t^k}\left[\frac{1}{\Gamma(k-\alpha)}\int_0^t (t-\lambda)^{k-\alpha-1} x^5 G(x\lambda^{-\frac{\alpha}{5}})\mathrm{d}\lambda\right].$$

Taking $\mu = \frac{t}{\lambda}$, above expression can be written in the following form:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^k}{\partial t^k}\left[\frac{x^5 t^{k-\alpha}}{\Gamma(k-\alpha)}\int_1^\infty (\mu-1)^{k-\alpha-1}\mu^{-(k-\alpha+1)}G(\theta\mu^{\frac{\alpha}{5}})\mathrm{d}\mu\right]\\ &= \frac{\partial^k}{\partial t^k}\left[x^5 t^{k-\alpha}\left(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G\right)(\theta)\right],\end{aligned} \tag{26}$$

where $(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G)(\theta)$ is the left-hand-sided Erdélyi-Kober fractional integral operator defined by (24). Further, for any differentiable function $\psi(\theta)$ where $\theta = xt^{-\frac{\alpha}{5}}$ the following must hold:

$$t\frac{\partial}{\partial t}\psi(\theta) = -\frac{\alpha}{5}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\psi(\theta).$$

The expression (26) can be simplified into the following:

$$\frac{\partial^k}{\partial t^k}\left[x^5 t^{k-\alpha}\left(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G\right)(\theta)\right] = \frac{\partial^{k-1}}{\partial t^{k-1}}\left[x^5 t^{k-\alpha-1}\left(k-\alpha-\frac{\alpha}{5}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\left(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G\right)(\theta)\right].$$

On repeating the above process $k-1$ times, we have

$$\frac{\partial^k}{\partial t^k}\left[x^5 t^{k-\alpha}\left(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G\right)(\theta)\right] = x^5 t^{-\alpha}\prod_{j=0}^{k-1}\left(1-\alpha+j-\frac{\alpha}{5}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\left(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G\right)(\theta).$$

By using the above into eq. (26), we obtain the following expression:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x^5 t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{5}{\alpha}}G\right)(\theta), \quad \forall\alpha > 0,$$

where $(\mathcal{P}^{1-\alpha,\alpha}_{\frac{5}{\alpha}}G)(\theta)$ is the left-hand sided Erdélyi-Kober fractional differential operator defined by (23). The coefficient functions $\beta(t)$, $\gamma(t)$, $\delta(t)$ are given by following relations:

$$\begin{aligned}\beta(t) &= k_1 t^{-\frac{7\alpha}{5}},\\ \gamma(t) &= k_2 t^{-\frac{7\alpha}{5}},\\ \delta(t) &= k_3 t^{-\frac{14\alpha}{5}},\end{aligned} \tag{27}$$

where k_1, k_2, k_3 are arbitrary constants.

The similarity variable and similarity transformation (25) along with coefficient functions (27) reduce eq. (1) into the following fractional ODE:

$$\begin{aligned}&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{5}{\alpha}}G\right)(\theta) + \theta^2G^2(\theta)(60k_1+100k_2) + \theta^3GG'(\theta)(60k_1+70k_2) + \theta^4GG''(\theta)(15k_1+5k_2)\\ &+ k_1\theta^5G'''(\theta) + 10k_2\theta^4{G'}^2(\theta) + k_2\theta^5G'G''(\theta) + 5k_3\theta^9G^3(\theta) + k_3\theta^{10}G^2G'\\ &+ \frac{120G(\theta)}{\theta^5} + \frac{600G'(\theta)}{\theta^4} + \frac{600G''(\theta)}{\theta^3} + \frac{200G'''(\theta)}{\theta^2} + \frac{25G''''(\theta)}{\theta} + G''''' = 0.\end{aligned} \tag{28}$$

To get a solution for eq. (1), let us assume that eq. (28) admits a solution in the form

$$G(\theta) = m_1\theta^{-7} + m_2\theta^{-2}. \tag{29}$$

On substituting this in (28), we arrive at a set of the following algebraic equations:

$$\begin{aligned}&\frac{\Gamma(1+\frac{2\alpha}{5})}{\Gamma(1-\frac{3\alpha}{5})}m_2 + (60k_1+100k_2)m_2^2 - 2(60k_1+70k_2)m_2^2 + 6(15k_1+5k_2)m_2^2 + 28k_2m_2^2 - 24k_1m_2^2 + 4k_3m_1m_2^2 = 0,\\ &(60k_1+100k_2)m_1^2 - 7(60k_1+70k_2)m_1^2 + 56(15k_1+5k_2)m_1^2 + 98k_2m_1^2 - 504k_1m_1^2 - 2k_3m_1^3 = 0 - 720a_1 = 0,\\ &\frac{\Gamma(1+\frac{7\alpha}{5})}{\Gamma(1+\frac{2\alpha}{5})}m_1 + 2(60k_1+100k_2)m_1m_2 - 9(60k_1+70k_2)m_1m_2 + 62(15k_1+5k_2)m_1m_2 + 126k_2m_1m_2\\ &- 528k_1m_1m_2 - k_3m_1^2m_2 = 0, \quad 3k_3m_2^3 = 0.\end{aligned} \tag{30}$$

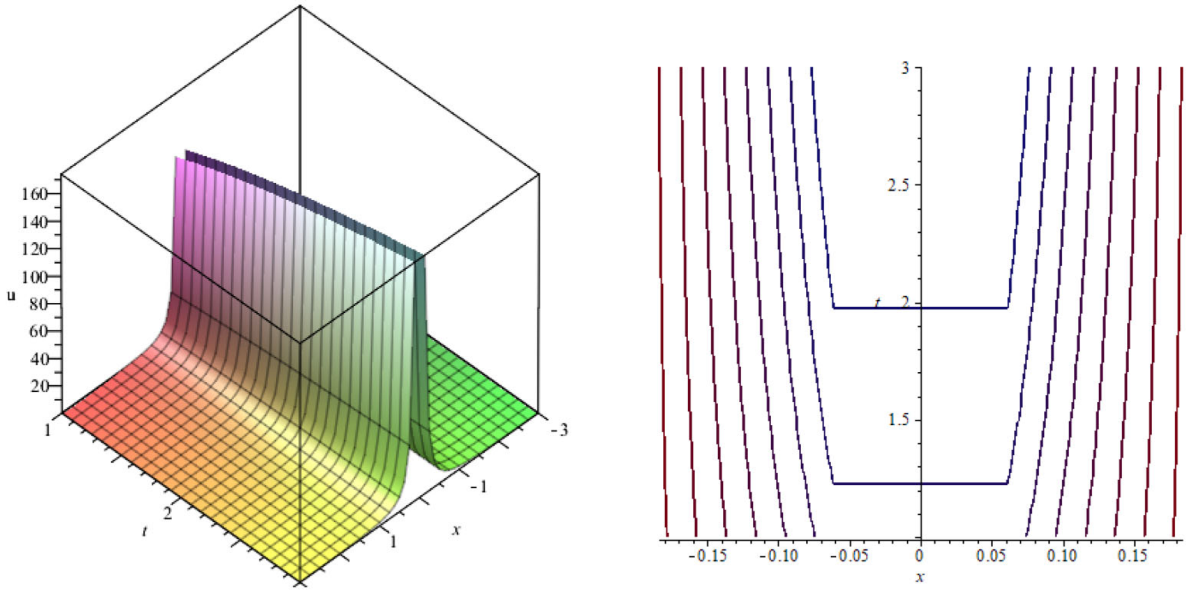

Fig. 1. Panels (a) and (b) represent the 3D plot and contour plot of solution (32), respectively, having parameter values $m_1 = 2$ and $\alpha = 0.2$.

After solving the above equations with the help of MAPLE software, we obtain the following solution:

$$\begin{aligned} m_1 &= m_1, \\ m_2 &= -\frac{m_1(\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{7}{5}\alpha) - \Gamma(1+\frac{2}{5}\alpha)^2)}{720(\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{2}{5}\alpha))}, \\ k_1 &= \frac{-12(3\,\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{7}{5}\alpha) - \Gamma(1+\frac{2}{5}\alpha)^2)}{m_1(\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{7}{5}\alpha) - \Gamma(1+\frac{2}{5}\alpha)^2)}, \\ k_2 &= \frac{12(\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{7}{5}\alpha) + 3\,\Gamma(1+\frac{2}{5}\alpha)^2)}{m_1(\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{7}{5}\alpha) - \Gamma(1+\frac{2}{5}\alpha)^2)}, \\ k_3 &= 0. \end{aligned} \tag{31}$$

Consequently, the solution of eq. (28) can be expressed as follows:

$$u = m_1 x^{-2} t^{\frac{7\alpha}{5}} - \frac{m_1(\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{7}{5}\alpha) - \Gamma(1+\frac{2}{5}\alpha)^2)}{720(\Gamma(1-\frac{3}{5}\alpha)\,\Gamma(1+\frac{2}{5}\alpha))} x^3 t^{\frac{2\alpha}{5}}. \tag{32}$$

From fig. 1, it is clear that the solution (32) is singular at $x = 0$ due to the negative power of x in term $x^{-2}t^{\frac{7\alpha}{5}}$. If we take positive values of x, the solution is concave upward and smooth and same is true for negative values of x.

Case ii) Generator Δ_2

We can go for reduction by solving the characteristics equations

$$\frac{\mathrm{d}x}{x/5} = \frac{\mathrm{d}t}{t/\alpha} = \frac{\mathrm{d}u}{0}.$$

After solving the characteristics equations, we find the similarity variable and reduction field as follows:

$$\theta = x t^{-\frac{\alpha}{5}}, \qquad u = G(\theta). \tag{33}$$

In this case, the Riemann-Liouville fractional derivative of $u = G(\theta)$ can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^k}{\partial t^k}\left[\frac{1}{\Gamma(k-\alpha)}\int_0^t (t-\lambda)^{k-\alpha-1} G(x\lambda^{-\frac{\alpha}{5}})\mathrm{d}\lambda\right]. \tag{34}$$

Taking $\mu = \frac{t}{\lambda}$, eq. (34) can be written as follows:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^k}{\partial t^k}\left[\frac{t^{k-\alpha}}{\Gamma(k-\alpha)}\int_1^\infty (\mu-1)^{k-\alpha-1}\mu^{-(k-\alpha+1)}G(\theta\mu^{\frac{\alpha}{5}})\mathrm{d}\mu\right]\\ &= \frac{\partial^k}{\partial t^k}\left[t^{k-\alpha}\left(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G\right)(\theta)\right]\\ &= t^{-\alpha}\prod_{j=0}^{k-1}\left(1-\alpha+j-\frac{\alpha}{5}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\left(\mathcal{K}^{1,k-\alpha}_{\frac{5}{\alpha}}G\right)(\theta)\\ &= t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{5}{\alpha}}G\right)(\theta). \end{aligned} \tag{35}$$

The coefficient functions $\beta(t)$, $\gamma(t)$, $\delta(t)$ become

$$\beta(t) = k_4 t^{-\frac{2}{5}\alpha}, \qquad \gamma(t) = k_5 t^{-\frac{2}{5}\alpha}, \qquad \delta(t) = k_6 t^{-\frac{4}{5}\alpha}. \tag{36}$$

The similarity transformation (33) along with eq. (36) reduce eq. (1) into the following fractional ordinary differential equation:

$$\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{5}{\alpha}}G\right)(\theta) + k_4 GG'''(\theta) + k_5 G'G''(\theta) + k_6 G^2G'(\theta) + G''''' = 0. \tag{37}$$

We investigate eq. (37) for power series solutions [13] of the form

$$G(\theta) = \sum_{n=0}^{\infty} a_n\theta^n. \tag{38}$$

Here a_n's are unknown coefficients that need to be determined later.

From eq. (38), we have

$$\begin{aligned} G'(\theta) &= \sum_{n=0}^{\infty}(n+1)a_{n+1}\theta^n, \qquad G''(\theta) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\theta^n,\\ G'''(\theta) &= \sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}\theta^n,\\ G'''''(\theta) &= \sum_{n=0}^{\infty}(n+5)(n+4)(n+3)(n+2)(n+1)a_{n+5}\theta^n. \end{aligned} \tag{39}$$

After substituting eqs. (38) and (39) into ODE (37), we get

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{\Gamma(1-\frac{n\alpha}{5})}{\Gamma(1-\alpha-\frac{n\alpha}{5})}a_n\theta^n + k_4\sum_{n=0}^{\infty}a_n\theta^n\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}\theta^n\\ &+ k_5\sum_{n=0}^{\infty}(n+1)a_{n+1}\theta^n\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\theta^n\\ &+ k_6\sum_{n=0}^{\infty}a_n\theta^n\sum_{n=0}^{\infty}a_n\theta^n\sum_{n=0}^{\infty}(n+1)a_{n+1}\theta^n + \sum_{n=0}^{\infty}(n+5)(n+4)(n+3)(n+2)(n+1)a_{n+5}\theta^n = 0. \end{aligned} \tag{40}$$

Comparing the coefficients in eq. (40) for $n=0$, we get

$$a_5 = -\frac{1}{120}\left(\frac{1}{\Gamma(1-\alpha)}a_0 + 6k_4a_0a_3 + 2k_5a_1a_2 + k_6a_0^2a_1\right). \tag{41}$$

When $n \geq 1$, we have

$$
\begin{aligned}
a_{n+5} = &-\frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)}\left(\frac{\Gamma(1-\frac{n\alpha}{5})}{\Gamma(1-\alpha-\frac{n\alpha}{5})}a_n\right.\\
&+k_4\sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)a_k a_{n+3-k}\\
&+k_5\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)a_{k+1}a_{n+2-k}\\
&\left.+k_6\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)a_j a_{k-j}a_{n+1-k}\right).
\end{aligned}
\tag{42}
$$

Therefore, the power series solution for eq. (37) can be obtained in the form:

$$
\begin{aligned}
G(\theta) = &\ a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4\\
&-\frac{1}{120}\left(\frac{1}{\Gamma(1-\alpha)}a_0 + 6k_4 a_0 a_3 + 2k_5 a_1 a_2 + k_6 a_0^2 a_1\right)\theta^5\\
&-\sum_{n=1}^{\infty}\frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)}\left(\frac{\Gamma(1-\frac{n\alpha}{5})}{\Gamma(1-\alpha-\frac{n\alpha}{5})}a_n\right.\\
&+k_4\sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)a_k a_{n+3-k}\\
&+k_5\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)a_{k+1}a_{n+2-k}\\
&\left.+k_6\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)a_j a_{k-j}a_{n+1-k}\right)\theta^{n+5}.
\end{aligned}
\tag{43}
$$

Thus with the help of similarity transformation (33), power series solution for fifth-order KDV eq. (1) can be obtained as follows:

$$
\begin{aligned}
u = &\ a_0 + a_1 x t^{-\frac{\alpha}{5}} + a_2 x^2 t^{-\frac{2\alpha}{5}} + a_3 x^3 t^{-\frac{3\alpha}{5}} + a_4 x^4 t^{-\frac{4\alpha}{5}}\\
&-\frac{1}{120}\left(\frac{1}{\Gamma(1-\alpha)}a_0 + 6k_4 a_0 a_3 + 2k_5 a_1 a_2 + k_6 a_0^2 a_1\right)x^5 t^{-\alpha}\\
&-\sum_{n=1}^{\infty}\frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)}\left(\frac{\Gamma(1-\frac{n\alpha}{5})}{\Gamma(1-\alpha-\frac{n\alpha}{5})}a_n\right.\\
&+k_4\sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)a_k a_{n+3-k}\\
&+k_5\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)a_{k+1}a_{n+2-k}\\
&\left.+k_6\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)a_j a_{k-j}a_{n+1-k}\right)x^{n+5}t^{-\frac{(n+5)\alpha}{5}}.
\end{aligned}
\tag{44}
$$

Here a_0, a_1, a_2, a_3, a_4 ought to be taken as arbitrary.

Case iii) Generator Δ_3

In this case, after solving characteristic equations, we obtain following similarity variable and similarity transformation:

$$\theta = t, \qquad u = G(t). \tag{45}$$

As one can see from eq. (45), u degenerate into functions of variable t alone, which may not correspond to a physically interesting case. Therefore we prefer not to study this case.

4 Conclusion

In this paper, we successfully derive explicit exact solutions for the time-fractional variable-coefficient fifth-order KDV equation (1). Using symmetries and the concept of optimal system, we find inequivalent reduced fractional ODEs and obtain explicit exact solutions. The graphical representation of the solution has been provided with the help of Maple.

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