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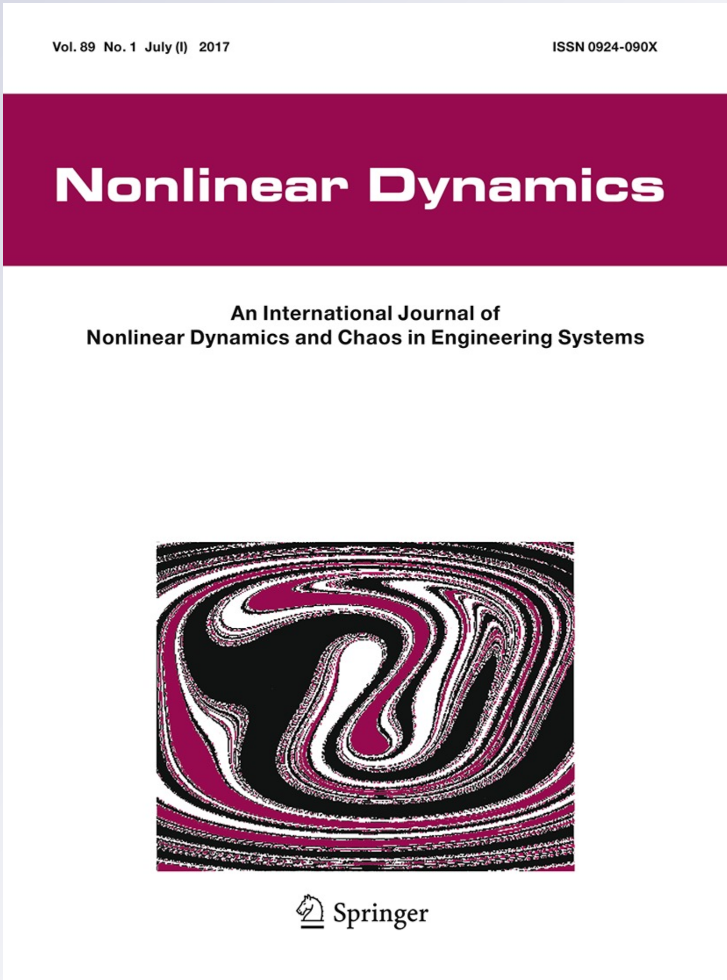
Nonlinear Dynamics
An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems

ISSN 0924-090X
Volume 89
Number 1

Nonlinear Dyn (2017) 89:321-331
DOI 10.1007/s11071-017-3456-7



Springer

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DOI 10.1007/s11071-017-3456-7

ORIGINAL PAPER

Space–time fractional nonlinear partial differential equations: symmetry analysis and conservation laws

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Received: 25 October 2016 / Accepted: 27 February 2017 / Published online: 14 March 2017

Abstract The symmetry method is developed to study space–time fractional nonlinear partial differential equations. Also, the Noether operators are extended for determining the conservation laws by application to some physically significant space–time fractional nonlinear partial differential equations.

Keywords Symmetry analysis · Space–time fractional partial differential equations · Erdélyi–Kober operators · Nonlinear self-adjointness · Conservation laws

Mathematics Subject Classification 26A33 · 34A08 · 35R11 · 76M60 · 70S10

1 Introduction

Fractional calculus developed as an exciting and one of the best tools for studying various models in science, engineering and mathematics. Many authors are devoted to the interpretation, properties and applications of fractional calculus [1–21]. The physical phenomena successfully modeled by using fractional calculus include viscoelasticity, fluid mechanics, anomalous diffusion, continuous-time random walk processes, long-range interactions, long-term behaviors, power laws, allometric scaling laws and fractal media [22–37]. Fractional differential equations (FDEs) have been increasingly studied by many researchers working in various fields of science [38–43]. A range of fractional partial differential equations (FPDEs) [44–50] with only time derivative of fractional order have been studied by using the Lie symmetry method. In our recent study [51], the symmetry approach has been extended for systems of time fractional partial differential equations (PDEs). But to the best of our knowledge, the symmetry method has not been introduced for analysis of the FPDEs with both space and time derivative of fractional order. However, a few authors [52] discussed the symmetry analysis of space–time fractional PDEs only considering their invariance under the Lie group of scaling transformations. In this study, the required prolongation formulae are derived to execute the complete group classification of the space–time fractional PDEs.

The symmetries and conservation laws provide a lot of information about the systems modeled by the differential equations. The symmetries are useful in determining exact solutions and conservation laws of differential equations. Despite the importance of conservation laws in investigating integrability, internal properties and proving existence and uniqueness of solutions

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of differential equations [53–55], the conservation laws for fractional-order PDEs are not widely discussed. The generalizations of Noether's theorem have been presented [56,57] to find conservation laws for FDEs. Recently, the fractional generalized Noether operators are proposed [55] for time fractional PDEs not having Lagrangians to find conservation laws using new conservation theorem [58]. Although a few works [59–61] dealing with conservation laws of time fractional PDEs can be noticed, the investigation of conserved vectors for the space–time fractional PDEs is completely unexplored. The main aim of this study is to provide the Lie symmetry method and generalization of Noether operators for symmetry analysis as well as conservation laws of the space–time fractional nonlinear PDEs.

The efficiency of the proposed approach is proved through following space–time fractional nonlinear PDEs:

1. The space–time fractional Gilson–Pickering (STFGP) equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a\frac{\partial^\beta u}{\partial x^\beta} - bu_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

2. The space–time fractional generalized Korteweg–de Vries (STFgKdV) equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^\beta u}{\partial x^\beta} + 6cu_x^3 + 18cuu_x u_{xx} + 3cu^2 u_{xxx} = 0, \tag{2}$$

where $0 < \alpha, \beta < 2$. The well-known KdV equation was derived from the propagation of dispersive shallow water waves and used for modeling many phenomena such as shock wave formation, solitons, turbulence and mass transport [62–65]. The time fractional KdV equation has been discussed widely in literature by several techniques [66–69]. Here, the fractional generalized KdV equation is considered [70,71] with $m = 1, n = 3$ and both space and time derivatives of fractional order. Also, the Gilson–Pickering equation has some important applications in nonlinear physics and has been studied for its behavior and exact traveling wave solutions [72–74]. In this work, the space–time fractional Gilson–Pickering equation is discussed.

The paper is organized as follows. In Sect. 2, the Lie symmetry method is developed for studying the space–time fractional PDEs. Section 3 deals with the symmetry analysis of the STFGP Eq. (1). Also, its nonlinear self-adjointness and conservation laws are discussed. Section 4 is devoted to the symmetry analysis, nonlinear self-adjointness and conservation laws of the STFgKdV Eq. (2). The last Sect. 5 consists of the conclusion of entire study.

2 Symmetry analysis for space–time fractional partial differential equations

Consider a space–time fractional PDE with two independent variables given in the following form:

$$F(x, t, u, \partial_t^\alpha u, \partial_x^\beta u, u_{xx}, u_{xxx}, \ldots) = 0, \quad \alpha > 0, \quad \beta > 0, \tag{3}$$

where subscripts denote the partial derivatives and fractional derivatives are considered in the Riemann–Liouville sense defined below:

Definition The Riemann–Liouville fractional derivative of a function $u(x, t)$, for $\alpha > 0$, is defined as follows [43,51]:

$$D_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(x,s)\mathrm{d}s, & n-1<\alpha<n \in \mathbb{N}, \\ \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\Gamma(z)$ is the standard Euler's Gamma function.

The admitted one parameter Lie group of transformations has the symmetry generator in the following form:

$$X = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u, \tag{5}$$

such that the prolonged generator can be defined as follows:

$$\mathrm{pr}^{(\alpha,\beta,i)} X = X + \eta^{\alpha,t}\partial_{\partial_t^\alpha u} + \eta^{\beta,x}\partial_{\partial_x^\beta u} + \eta^{2x}\partial_{u_{2x}} + \ldots + \eta^{ix}\partial_{u_{ix}}, \tag{6}$$

where i is the order of the FPDE (3) and $u_{ix} = \frac{\partial^i u}{\partial x^i}$. The operators η^{jx} are jth ($j = 2, 3, ..$)-order extended symmetry operators [53] and $(\eta^{\alpha,t}, \eta^{\beta,x})$ are fractional extended operators defined as follows:

$$\begin{aligned}\eta^{\alpha,t} &= D_t^{\alpha}(\eta) + \xi D_t^{\alpha}(u_x) - D_t^{\alpha}(\xi u_x) + D_t^{\alpha}(u(D_t\tau)) \\ &\quad - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \\ \eta^{\beta,x} &= D_x^{\beta}(\eta) + D_x^{\beta}(u(D_x\xi)) - D_x^{\beta+1}(\xi u) + \xi D_x^{\beta+1}(u) \\ &\quad + \tau D_x^{\beta}(u_t) - D_x^{\beta}(\tau u_t), \\ \eta^{xx} &= \mathrm{D_x}(\eta^{x}) - u_{xx}\mathrm{D_x}(\xi) - u_{xt}\mathrm{D_x}(\tau), \\ \eta^{xxx} &= \mathrm{D_x}(\eta^{xx}) - u_{xxx}\mathrm{D_x}(\xi) - u_{xxt}\mathrm{D_x}(\tau), \\ &\vdots \end{aligned} \tag{7}$$

where the symbols $\mathrm{D_t}, \mathrm{D_x}$ denote the total derivative operators defined by

$$\begin{aligned}\mathrm{D_t} &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{xt}\partial_{u_x} + \cdots, \\ \mathrm{D_x} &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t} + \cdots. \end{aligned} \tag{8}$$

In view of the generalized Leibnitz rule [43] and the generalized chain rule [43,75], direct calculation implies the extended symmetry operator $\eta^{\beta,x}$ can be introduced in the following form:

$$\begin{aligned}\eta^{\beta,x} &= \partial_x^{\beta}\eta + (\eta_u - \beta\mathrm{D_x}(\xi))\,\partial_x^{\beta}u - u\partial_x^{\beta}\eta_u \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{\beta}{n}\partial_x^{n}\eta_u - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}(\xi)\right]D_x^{\beta-n}(u) \\ &\quad - \sum_{n=1}^{\infty}\binom{\beta}{n}\mathrm{D}_x^{n}(\tau)D_x^{\beta-n}(u_t) + \mu_{\beta}, \end{aligned} \tag{9}$$

where

$$\begin{aligned}\mu_{\beta} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{x^{n-\beta}}{\Gamma(n-\beta+1)}\right. \\ &\quad \left.\times(-u)^{r}\frac{\partial^{m}}{\partial x^{m}}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial u^{k}}\right]. \end{aligned} \tag{10}$$

Equivalently, the αth-order extended infinitesimal $\eta^{\alpha,t}$ can be written as follows [44–46]:

$$\begin{aligned}\eta^{\alpha,t} &= \partial_t^{\alpha}\eta + (\eta_u - \alpha\mathrm{D_t}(\tau))\partial_t^{\alpha}u - u\partial_t^{\alpha}\eta_u \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^{n}\eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) \\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathrm{D}_t^{n}(\xi)D_t^{\alpha-n}(u_x) + \mu_{\alpha}, \end{aligned} \tag{11}$$

where

$$\begin{aligned}\mu_{\alpha} &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}\right. \\ &\quad \left.\times(-u)^{r}\frac{\partial^{m}}{\partial t^{m}}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^{k}}\right]. \end{aligned} \tag{12}$$

The invariance criterion for the FPDE (3) can be written as follows:

$$\mathrm{pr}^{(\alpha,\beta,i)}X(\Delta)\Big|_{\Delta=0} = 0, \tag{13}$$

where $\Delta = F(x, t, u, \partial_t^{\alpha}u, \partial_x^{\beta}u, u_{xx}, u_{xxx}, \ldots)$.

Hence, using this extended approach, the symmetry analysis of the space–time fractional PDEs can be easily investigated.

3 The STFGP equation

In this section, the STFGP Eq. (1) is considered for its symmetry analysis followed by the investigation of conserved vectors.

3.1 Symmetry analysis

For generator X given by (5), the third-order prolongation $\mathrm{pr}^{(\alpha,\beta,3)}X$ for FPDE (1) gives the following invariance criterion:

$$\begin{aligned}&[\eta^{\alpha,t} + a\eta^{\beta,x} - b\eta^{x}u_{xx} - bu_x\eta^{xx} \\ &\quad -\eta u_{xxx} - u\eta^{xxx}]\Big|_{(1)} = 0. \end{aligned} \tag{14}$$

Substituting the extended infinitesimals and equating the coefficients of alike partial derivatives, fractional derivatives and powers of u, the set of determining equations can be obtained as follows:

$$\begin{aligned}&\xi_t = \xi_u = 0, \\ &\tau_x = \tau_u = 0, \\ &\alpha\tau_t - \beta\xi_x = 0, \\ &\eta_{uu} = 0, \\ &\eta = u(3\xi_x - \alpha\tau_t), \\ &\binom{\alpha}{n}\partial_t^{n}\eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad \forall n\in\mathbb{N}, \\ &\binom{\beta}{n}\partial_x^{n}\eta_u - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}\xi = 0, \quad \forall n\in\mathbb{N}, \\ &\partial_t^{\alpha}\eta - u\partial_t^{\alpha}\eta_u + a(\partial_x^{\beta}\eta - u\partial_x^{\beta}\eta_u) - u\eta_{xxx} = 0. \end{aligned} \tag{15}$$

Solving these equations, the resulting group infinitesimals are as follows:

$$\xi = \frac{c_1 x}{\beta} + c_2, \quad \tau = \frac{c_1 t}{\alpha} + c_3, \quad \eta = c_1 u\left(\frac{3-\beta}{\beta}\right), \tag{16}$$

where c_1, c_2 and c_3 are arbitrary constants. Since there are fractional derivatives with respect to both x and t, the invariance of the fixed lower limit of the integral of the type (4) gives the following conditions:

$$\xi(x,t,u)|_{x=0}=0, \quad \tau(x,t,u)|_{t=0}=0. \tag{17}$$

Thus, the corresponding infinitesimal generator can be written as follows:

$$X=\frac{x}{\beta}\frac{\partial}{\partial x}+\frac{t}{\alpha}\frac{\partial}{\partial t}+u\left(\frac{3-\beta}{\beta}\right)\frac{\partial}{\partial u}. \tag{18}$$

Solving the associated characteristic equations gives the following similarity solutions:

$$z=xt^{-\frac{\alpha}{\beta}}, \quad u=t^{\frac{\alpha(3-\beta)}{\beta}}F. \tag{19}$$

Before finding the symmetry reductions, let us introduce the left-hand-sided Erdélyi–Kober fractional differential operator [31,52] in the following form:

$$\left(\mathcal{P}_{\delta}^{\zeta,\alpha}h\right)(z):=\left(\prod_{j=0}^{m-1}\left(\zeta+j-\frac{1}{\delta}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)(\mathcal{K}_{\delta}^{\zeta+\alpha,m-\alpha}h)(z),$$
$$m=\begin{cases}[\alpha]+1 & \text{if } \alpha\notin\mathbb{N},\\ \alpha & \text{if } \alpha\in\mathbb{N},\end{cases}\quad z>0,\ \delta>0,\ \alpha>0, \tag{20}$$

where

$$\left(\mathcal{K}_{\delta}^{\zeta,\alpha}h\right)(z):=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^{\infty}(s-1)^{\alpha-1}s^{-(\zeta+\alpha)}h(zs^{\frac{1}{\delta}})\mathrm{d}s & \text{if } \alpha>0,\\ h(z) & \text{if } \alpha=0,\end{cases} \tag{21}$$

is the left-hand-sided Erdélyi–Kober fractional integral operator. Also, we introduce the right-hand-sided Erdélyi–Kober fractional differential operator [31,52] as follows:

$$\left(\mathcal{D}_{\delta}^{\zeta,\beta}h\right)(z):=\left(\prod_{j=1}^{m}\left(\zeta+j+\frac{1}{\delta}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)(\mathcal{I}_{\delta}^{\zeta+\beta,m-\beta}h)(z),$$
$$m=\begin{cases}[\beta]+1 & \text{if } \beta\notin\mathbb{N},\\ \beta & \text{if } \beta\in\mathbb{N},\end{cases}\quad z>0,\ \delta>0,\ \alpha>0, \tag{22}$$

where

$$\left(\mathcal{I}_{\delta}^{\zeta,\beta}h\right)(z):=\begin{cases}\frac{1}{\Gamma(\beta)}\int_0^{1}(1-s)^{\beta-1}s^{\zeta}h(zs^{\frac{1}{\delta}})\mathrm{d}s & \text{if } \beta>0,\\ h(z) & \text{if } \beta=0,\end{cases} \tag{23}$$

is the right-hand-sided Erdélyi–Kober fractional integral operator.

By the definition of Riemann–Liouville fractional derivative, the αth-order fractional derivative of $u(x,t)=t^{\frac{\alpha(3-\beta)}{\beta}}F(z)$ with respect to t for $n-1<\alpha<n$ ($n\in\mathbb{N}$) is given by

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}s^{\frac{\alpha(3-\beta)}{\beta}}F(xs^{-\frac{\alpha}{\beta}})\mathrm{d}s\right]. \tag{24}$$

Taking $w=\frac{t}{s}$, it can be written in the following form:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\alpha+\frac{\alpha(3-\beta)}{\beta}}}{\Gamma(n-\alpha)}\int_1^{\infty}(w-1)^{n-\alpha-1}\times w^{-(n-\alpha+1+\frac{\alpha(3-\beta)}{\beta})}F(zw^{\frac{\alpha}{\beta}})\mathrm{d}w\right],$$
$$=\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha+\frac{\alpha(3-\beta)}{\beta}}\left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(3-\beta)}{\beta},n-\alpha}F\right)(z)\right], \tag{25}$$

where the operator $\left(\mathcal{K}_{\delta}^{\zeta,\alpha}\right)$ is defined by (21). The relation (25) is also true for $\alpha=n=1,2,3,\ldots$ because $\left(\mathcal{K}_{\delta}^{\zeta,0}F\right)(z)=F(z)$. For $z=xt^{-\frac{\alpha}{\beta}}$ and a function $\psi(z)\in C^1(0,\infty)$, the following relation holds:

$$t\frac{\mathrm{d}}{\mathrm{d}t}\psi(z)=-\frac{\alpha}{\beta}z\frac{\mathrm{d}}{\mathrm{d}z}\psi(z). \tag{26}$$

It follows that (25) can be written as follows:

$$\begin{aligned}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-1+\frac{\alpha(3-\beta)}{\beta}}\times\left(n-\alpha+\frac{\alpha(3-\beta)}{\beta}-\frac{\alpha}{\beta}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(3-\beta)}{\beta},n-\alpha}F\right)(z)\right],\\ &=\cdots\\ &=t^{-\alpha+\alpha\left(\frac{3-\beta}{\beta}\right)}\prod_{j=0}^{n-1}\left(1-\alpha+\alpha\left(\frac{3-\beta}{\beta}\right)+j-\frac{\alpha}{\beta}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\times\left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(3-\beta)}{\beta},n-\alpha}F\right)(z),\\ &=t^{-\alpha+\alpha\left(\frac{3-\beta}{\beta}\right)}\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha(3-\beta)}{\beta},\alpha}F\right)(z),\qquad \forall\alpha>0,\end{aligned} \tag{27}$$

where $\left(\mathcal{P}_{\delta}^{\zeta,\alpha}\right)$ is the left-hand-sided Erdélyi–Kober fractional differential operator defined by (20).

In the similar manner, the βth-order Riemann–Liouville fractional derivative with respect to x can be given as follows:

$$\frac{\partial^\beta u}{\partial x^\beta} = t^{\frac{\alpha(3-\beta)}{\beta}} x^{-\beta} \left(\mathcal{D}_1^{-\beta,\beta} F\right)(z), \qquad \forall \beta > 0, \quad (28)$$

where $\left(\mathcal{D}_\delta^{\zeta,\beta}\right)$ is the right-hand-sided Erdélyi–Kober fractional differential operator defined by (22).

Hence, the STFGP Eq. (1) is reduced to the nonlinear fractional ordinary differential equation (FODE) for $\alpha, \beta > 0$ written as follows:

$$\begin{aligned}&\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha}{\beta}(3-\beta),\alpha} F\right)(z) + az^{-\beta}\left(\mathcal{D}_1^{-\beta,\beta} F\right)(z) \\ &\quad - bF'(z)F''(z) - FF'''(z) = 0.\end{aligned} \quad (29)$$

Next step is to find the conserved vectors with the help of the obtained Lie symmetries. Before that the nonlinear self-adjointness of the STFGP Eq. (1) is investigated in next subsection.

3.2 Nonlinear self-adjointness

The concept of nonlinear self-adjointness [76] for integer-order PDEs was proposed for calculating the conservation laws by using new conservation theorem [58]. Recently, the nonlinear self-adjointness of a few time fractional PDEs [55,60,61] has been studied. Here, this concept is extended to the space–time fractional PDEs by its application to the STFGP Eq. (1). A formal Lagrangian [58] for the FPDE (1) is given by

$$\mathcal{L} = v(x,t)(\partial_t^\alpha u + a\partial_x^\beta u - bu_x u_{xx} - uu_{xxx}), \quad (30)$$

where $v(x,t)$ is a new dependent variable. The adjoint equation of the STFGP Eq. (1) is defined by

$$F^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \quad (31)$$

where $\frac{\delta}{\delta u}$ is the Euler–Lagrange operator defined as follows:

$$\begin{aligned}\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial(D_t^\alpha u)} + (D_x^\beta)^* \frac{\partial}{\partial(D_x^\beta u)} \\ &\quad + \sum_{k=1}^{\infty}(-1)^k D_{i_1} D_{i_2}, \ldots, D_{i_k} \frac{\partial}{\partial u_{i_1 i_2,\ldots,i_k}}.\end{aligned} \quad (32)$$

Here $(D_t^\alpha)^*$ and $(D_x^\beta)^*$ are the adjoint operators of Riemann–Liouville fractional differential operators D_t^α and D_x^β, respectively, given as follows:

$$\begin{aligned}(D_t^\alpha)^* &= (-1)^n I_p^{n-\alpha}(D_t^n) = {}_t^C D_p^\alpha, \\ (D_x^\beta)^* &= (-1)^m I_q^{m-\beta}(D_x^m) = {}_x^C D_q^\beta,\end{aligned} \quad (33)$$

where $I_p^{n-\alpha}$ and $I_q^{m-\beta}$ are the right-hand-sided fractional integral operators of order $n-\alpha$ and $m-\beta$, respectively, defined as follows:

$$I_p^{n-\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_t^p \frac{f(x,s)}{(s-t)^{1+\alpha-n}} \mathrm{d}s, \quad (34)$$

where $n = [\alpha] + 1$,

$$I_q^{m-\beta} f(x,t) = \frac{1}{\Gamma(m-\beta)} \int_x^q \frac{f(s,t)}{(s-x)^{1+\beta-m}} \mathrm{d}s, \quad (35)$$

where $m = [\beta] + 1$.

Also ${}_t^C D_p^\alpha$, ${}_x^C D_q^\beta$ are the right-hand-sided Caputo fractional differential operators [43] of order α and β, respectively.

According to (30) and (31), the adjoint equation of FPDE (1) can be obtained as follows:

$$\begin{aligned}F^* &\equiv (D_t^\alpha)^* v + a(D_x^\beta)^* v \\ &\quad + (3-b)(u_x v_{xx} + u_{xx} v_x) + uv_{xxx} = 0.\end{aligned} \quad (36)$$

For nonlinear self-adjointness of Eq. (1), the Eq. (36) should be satisfied for all the solutions of Eq. (1) with the following substitution

$$v = \phi(x,t,u), \qquad \text{where} \quad \phi(x,t,u) \neq 0. \quad (37)$$

The derivatives of (37) are given by

$$\begin{aligned}v_x &= \phi_x + \phi_u u_x, \\ v_{xx} &= \phi_{xx} + 2\phi_{xu} u_x + \phi_u u_{xx} + \phi_{uu} u_x^2, \\ v_{xxx} &= \phi_{xxx} + 3\phi_{xxu} u_x + 3\phi_{xuu} u_x^2 + 3\phi_{xu} u_{xx} \\ &\quad + 3\phi_{uu} u_x u_{xx} + \phi_u u_{xxx} + \phi_{uuu} u_x^3.\end{aligned} \quad (38)$$

Substituting (37) and (38) in the expression (36) gives the following condition for nonlinear self-adjointness:

$$\begin{aligned}&(D_t^\alpha)^*\phi + a(D_x^\beta)^*\phi + (3-b)\left(\phi_x u_{xx} + \phi_{xx} u_x\right. \\ &\quad \left. + 2\phi_u u_x u_{xx} + 2\phi_{xu} u_x^2 + \phi_{uu} u_x^3\right) \\ &\quad + u\left(\phi_{xxx} + 3\phi_{xxu} u_x + 3\phi_{xuu} u_x^2 + 3\phi_{xu} u_{xx}\right. \\ &\quad \left. + 3\phi_{uu} u_x u_{xx} + \phi_u u_{xxx} + \phi_{uuu} u_x^3\right) \\ &\quad = \lambda(\partial_t^\alpha u + a\partial_x^\beta u - bu_x u_{xx} - uu_{xxx}),\end{aligned} \quad (39)$$

with regular undetermined coefficient λ. Comparing the coefficients and solving includes the following condition:

$$(b-3)\phi_x = 0, \quad (40)$$

which splits into the following two cases:

Case 1 $\mathbf{b = 3}$
In this case, the solution of (39) can be obtained as follows:

$$\phi = A_1(t)x^2 + B_1(t)x + C_1(t) \quad \text{and} \quad \lambda = 0, \tag{41}$$

where $A_1(t)$, $B_1(t)$, $C_1(t)$ are arbitrary functions of t such that the following holds:

$$x^2\left({}_t^C D_p^\alpha A_1(t)\right) + x\left({}_t^C D_p^\alpha B_1(t)\right) + \left({}_t^C D_p^\alpha C_1(t)\right) + a\left[A_1(t)\left({}_x^C D_q^\beta x^2\right) + B_1(t)\left({}_x^C D_q^\beta x\right)\right] = 0. \tag{42}$$

The values of $\left({}_x^C D_q^\beta x\right)$ and $\left({}_x^C D_q^\beta x^2\right)$ depend on β leading to the subcases given below:

Subcase $\mathbf{0 < \beta < 1}$
In this subcase, using the values of right-hand-sided Caputo fractional derivatives $\left({}_x^C D_q^\beta x\right)$ and $\left({}_x^C D_q^\beta x^2\right)$ in (42) gives the following solution:

$$A_1(t) = 0 = B_1(t), \quad C_1(t) = a_1, \tag{43}$$

where a_1 is an arbitrary constant. Then from (37), the result is of the form:

$$v = a_1. \tag{44}$$

Subcase $\mathbf{1 < \beta < 2}$
In this subcase, using the values of $\left({}_x^C D_q^\beta x\right)$ and $\left({}_x^C D_q^\beta x^2\right)$ in (42) implies the following:

$$A_1(t) = 0, \quad B_1(t) = a_2, \quad C_1(t) = a_3, \tag{45}$$

such that the following solution of $v(x, t)$ is obtained:

$$v = a_2 x + a_3, \tag{46}$$

where a_2 and a_3 are arbitrary constants.

Case 2 $\mathbf{b \neq 3}$, $\phi_\mathbf{x} = \mathbf{0}$
In this case, solution $v(x, t)$ is obtained as follows:

$$v = a_4, \tag{47}$$

where a_4 is an arbitrary constant.

These solutions of $v(x, t)$ are substituted in the formal Lagrangian (30) for the construction of conserved vectors in next subsection.

3.3 Conservation laws

In this section, the conservation laws for the STFGP Eq. (1) are obtained using the Lie symmetry generator (18). It is well known that a vector (C^t, C^x) that satisfies the equation given by

$$D_t(C^t) + D_x(C^x)|_{(1)} = 0, \tag{48}$$

is called the conserved vector. The existence of fractional derivatives of both independent variables x and t indicates the requirement of the fractional generalization of the Noether operators. The fractional Noether operator for the variable t is given by [55,59,60]

$$C^t = \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W) D_t^k\left(\frac{\partial\mathcal{L}}{\partial(D_t^\alpha u)}\right) - (-1)^n J\left(W, D_t^n\left(\frac{\partial\mathcal{L}}{\partial(D_t^\alpha u)}\right)\right), \tag{49}$$

where $n = [\alpha] + 1$, $W = \eta - \xi u_x - \tau u_t$ is the Lie characteristic function for generator $X = \xi\partial_x + \tau\partial_t + \eta\partial_u$ and J is the integral defined by

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^p \frac{f(x, s)g(x, r)}{(r-s)^{\alpha+1-n}} \mathrm{d}r\mathrm{d}s, \tag{50}$$

for any two functions $f(x, t)$ and $g(x, t)$.
Equivalently, the fractional Noether operator for the component x of the conserved vector is introduced as follows:

$$C^x = \sum_{k=0}^{m-1}(-1)^k D_x^{\beta-1-k}(W) D_x^k\left(\frac{\partial\mathcal{L}}{\partial(D_x^\beta u)}\right) - (-1)^n J_1\left(W, D_x^m\left(\frac{\partial\mathcal{L}}{\partial(D_x^\beta u)}\right)\right), \tag{51}$$

where $m = [\beta] + 1$ and J_1 is the integral defined by

$$J_1(f, g) = \frac{1}{\Gamma(m-\beta)}\int_0^x\int_x^q \frac{f(s, t)g(r, t)}{(r-s)^{\beta+1-m}} \mathrm{d}r\mathrm{d}s. \tag{52}$$

Here, for the symmetry X given by (18), the function W is of the form

$$W = u\left(\frac{3-\beta}{\beta}\right) - \frac{x}{\beta}u_x - \frac{t}{\alpha}u_t. \tag{53}$$

Firstly, the conservation laws are obtained for the solution (44) of variable $v(x, t)$. Substituting the Lagrangian (30) in (51) using (44) with $a_1 = 1$, the x component of the conserved vector can be obtained as follows:

$$C^x = a\left[\left(\frac{3-\beta}{\beta}\right) I_x^{1-\beta}(u) - \frac{1}{\beta} I_x^{1-\beta}(x u_x) - \frac{t}{\alpha} I_x^{1-\beta}(u_t)\right]. \tag{54}$$

Also, the t component of the conserved vector is obtained in the following form:

case $0 < \alpha < 1$

$$C^t = \left[\left(\frac{3-\beta}{\beta}\right) I_t^{1-\alpha}(u) - \frac{x}{\beta} I_t^{1-\alpha}(u_x) - \frac{1}{\alpha} I_t^{1-\alpha}(tu_t)\right]. \tag{55}$$

case $1 < \alpha < 2$

$$C^t = \left[\left(\frac{3-\beta}{\beta}\right) D_t^{\alpha-1}(u) - \frac{x}{\beta} D_t^{\alpha-1}(u_x) - \frac{1}{\alpha} D_t^{\alpha-1}(tu_t)\right]. \tag{56}$$

Note that the conservation laws with the fractional components C^x and C^t are obtained in terms of Riemann–Liouville fractional derivative operators and do not involve the integrals (50) and (52). The fractional conserved vectors for the remaining solutions of $v(x,t)$ are calculated as follows:

In case of (46), the x component of the conserved vector is as follows:

$$\begin{aligned} C^x &= a(a_2x + a_3) \\ &\times \left[\left(\frac{3-\beta}{\beta}\right) D_x^{\beta-1}(u) - \frac{1}{\beta} D_x^{\beta-1}(xu_x) - \frac{t}{\alpha} D_x^{\beta-1}(u_t)\right] \\ &-aa_2\left[\left(\frac{3-\beta}{\beta}\right) I_x^{2-\beta}(u) - \frac{1}{\beta} I_x^{2-\beta}(xu_x) - \frac{t}{\alpha} I_x^{2-\beta}(u_t)\right]. \end{aligned} \tag{57}$$

Thus, corresponding to the constants a_2 and a_3, the linearly independent components of the conserved vector with respect to x are given as follows:

$$\begin{aligned} C_{a_2}^x &= ax\left[\left(\frac{3-\beta}{\beta}\right) D_x^{\beta-1}(u) - \frac{1}{\beta} D_x^{\beta-1}(xu_x) - \frac{t}{\alpha} D_x^{\beta-1}(u_t)\right] \\ &-a\left[\left(\frac{3-\beta}{\beta}\right) I_x^{2-\beta}(u) - \frac{1}{\beta} I_x^{2-\beta}(xu_x) - \frac{t}{\alpha} I_x^{2-\beta}(u_t)\right], \end{aligned} \tag{58}$$

and

$$\begin{aligned} C_{a_3}^x &= a\left[\left(\frac{3-\beta}{\beta}\right) D_x^{\beta-1}(u) \right.\\ &\left. -\frac{1}{\beta} D_x^{\beta-1}(xu_x) - \frac{t}{\alpha} D_x^{\beta-1}(u_t)\right]. \end{aligned} \tag{59}$$

For $0 < \alpha < 1$, the t component of conserved vector for constant a_2 is of the following form:

$$C_{a_2}^t = x\left[\left(\frac{3-\beta}{\beta}\right) I_t^{1-\alpha}(u) - \frac{x}{\beta} I_t^{1-\alpha}(u_x) - \frac{1}{\alpha} I_t^{1-\alpha}(tu_t)\right], \tag{60}$$

and the component $C_{a_3}^t$ coincides with vector (55).

For $1 < \alpha < 2$, the t component for a_2 is obtained as follows:

$$C_{a_2}^t = x\left[\left(\frac{3-\beta}{\beta}\right) D_t^{\alpha-1}(u) - \frac{x}{\beta} D_t^{\alpha-1}(u_x) - \frac{1}{\alpha} D_t^{\alpha-1}(tu_t)\right], \tag{61}$$

and the component $C_{a_3}^t$ is same as vector (56).

In case of (47), the conserved vector has the following components:

case $0 < \beta < 1$

The obtained x component of conserved vector is coincident with (54).

case $1 < \beta < 2$

In this case, the x component of the conserved vector coincides with vector (59).

case $0 < \alpha < 1$

The t component of conserved vector in this case is coincident with (55).

case $1 < \alpha < 2$

The obtained t component coincides with vector (56).

4 The STFgKdV equation

In this section, the Lie symmetries and conservation laws of the STFgKdV Eq. (2) are calculated systematically.

4.1 Symmetry analysis

The invariance of the FPDE (2) under the admitted Lie group of transformations gives the following criterion:

$$\begin{aligned} &\left[\eta^{\alpha,t} + \eta^{\beta,x} + 18cu_x^2\eta^x + 18c\eta u_x u_{xx} + 18cuu_{xx}\eta^x \right.\\ &\left.\left. +18c\eta^{xx}uu_x + 6c\eta uu_{xxx} + 3cu^2\eta^{xxx}\right]\right|_{(2)} = 0. \end{aligned} \tag{62}$$

Inserting the extended symmetry operators and equating the coefficients, the set of determining equations can be obtained. Solving the determining equations, the infinitesimals can be derived in the following form:

$$\xi = \frac{c_1 x}{\beta}, \quad \tau = \frac{c_1 t}{\alpha}, \quad \eta = \frac{c_1 u}{2}\left(\frac{3}{\beta} - 1\right), \tag{63}$$

where c_1 is an arbitrary constant. The corresponding symmetry generator is given by

$$X = \frac{x}{\beta}\partial_x + \frac{t}{\alpha}\partial_t + \frac{u(3-\beta)}{2\beta}\partial_u, \tag{64}$$

such that the symmetry invariants can be easily calculated as follows:

$$z = xt^{\frac{-\alpha}{\beta}}, \quad u = t^{\frac{\alpha(3-\beta)}{2\beta}} F. \tag{65}$$

The above invariants are used for the reduction of the FPDE (2) into a nonlinear FODE given as follows:

$$\begin{aligned}&\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha}{2\beta}(3-\beta),\alpha} F\right)(z) + z^{-\beta}\left(\mathcal{D}_1^{-\beta,\beta} F\right)(z)\\ &\quad + 6c(F'(z))^3 + 18cFF'(z)\\ &F''(z) + 3cF^2F'''(z) = 0,\end{aligned} \tag{66}$$

where $\left(\mathcal{P}_\delta^{\zeta,\alpha}\right)$ and $\left(\mathcal{D}_\delta^{\zeta,\beta}\right)$ are the left and right-hand-sided Erdélyi–Kober fractional differential operators defined by (20) and (22), respectively.

4.2 Nonlinear self-adjointness

A formal Lagrangian for the FPDE (2) can be written as follows:

$$\begin{aligned}\mathcal{L} = v(x,t)(&\partial_t^\alpha u + \partial_x^\beta u + 6cu_x^3\\ &+18cuu_xu_{xx} + 3cu^2u_{xxx}).\end{aligned} \tag{67}$$

In view of (31) and (67), the adjoint equation for FPDE (2) can be calculated as follows:

$$F^* \equiv (D_t^\alpha)^*v + (D_x^\beta)^*v - 3cu^2v_{xxx} = 0. \tag{68}$$

For nonlinear self-adjointness, assume the value of v given by

$$v = \phi(x,t,u), \quad \text{where} \ \ \phi(x,t,u) \neq 0. \tag{69}$$

Substituting (69) and its derivatives in (68) gives the following condition for nonlinear self-adjointness:

$$\begin{aligned}&(D_t^\alpha)^*\phi + (D_x^\beta)^*\phi - 3cu^2\\ &\times\Big(\phi_{xxx} + 3\phi_{xxu}u_x + 3\phi_{xuu}u_x^2\\ &\qquad + 3\phi_{xu}u_{xx} + 3\phi_{uu}u_xu_{xx} + \phi_uu_{xxx} + \phi_{uuu}u_x^3\Big)\\ &= \lambda(\partial_t^\alpha u + \partial_x^\beta u + 6cu_x^3 + 18cuu_xu_{xx} + 3cu^2u_{xxx}).\end{aligned} \tag{70}$$

Equating the coefficients and solving (70) leads to the following two cases:

case $0<\beta<1$: In this case, the value of ϕ and hence v is attained as below:

$$v = a_5, \tag{71}$$

where a_5 is an arbitrary constant.

case $1<\beta<2$: In this case, v can be obtained as follows:

$$v = a_6x + a_7, \tag{72}$$

where a_6 and a_7 are arbitrary constants.

These values of variable $v(x,t)$ are used for investigating conservation laws for the FPDE (2).

4.3 Conservation laws

For the symmetry generator X given by (64), we have the following:

$$W = \frac{u}{2}\left(\frac{3-\beta}{\beta}\right) - \frac{x}{\beta}u_x - \frac{t}{\alpha}u_t. \tag{73}$$

For case (71), the x component of the conserved vector can be calculated in the following form:

$$C^x = \left(\frac{3-\beta}{2\beta}\right) I_x^{1-\beta}(u) - \frac{1}{\beta}I_x^{1-\beta}(xu_x) - \frac{t}{\alpha}I_x^{1-\beta}(u_t). \tag{74}$$

The t component of the conserved vector is given in following cases:

Case $0<\alpha<1$:

$$C^t = \left(\frac{3-\beta}{2\beta}\right) I_t^{1-\alpha}(u) - \frac{x}{\beta}I_t^{1-\alpha}(u_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tu_t). \tag{75}$$

Case $1<\alpha<2$:

$$C^t = \left(\frac{3-\beta}{2\beta}\right) D_t^{\alpha-1}(u) - \frac{x}{\beta}D_t^{\alpha-1}(u_x) - \frac{1}{\alpha}D_t^{\alpha-1}(tu_t). \tag{76}$$

For case (72), the two independent values of v are given by

$$v = 1, \quad v = x. \tag{77}$$

Firstly, in case of $v = 1$, the component of conserved vector with respect to x can be obtained as follows:

$$C^x = \left(\frac{3-\beta}{2\beta}\right) D_x^{\beta-1}(u) - \frac{1}{\beta}D_x^{\beta-1}(xu_x) - \frac{t}{\alpha}D_x^{\beta-1}(u_t). \tag{78}$$

where for $0 < \alpha < 1$, the conserved vector component for t coincides with (75) and for $1 < \alpha < 2$, the t component of conserved vector coincides with (76). In case of $v = x$, the x component of the conserved vector results in the following form:

$$C^x = x\left[\left(\frac{3-\beta}{2\beta}\right)D_x^{\beta-1}(u) - \frac{1}{\beta}D_x^{\beta-1}(xu_x) - \frac{t}{\alpha}D_x^{\beta-1}(u_t)\right] - \left[\left(\frac{3-\beta}{2\beta}\right)I_x^{2-\beta}(u) - \frac{1}{\beta}I_x^{2-\beta}(xu_x) - \frac{t}{\alpha}I_x^{2-\beta}(u_t)\right]. \tag{79}$$

In this case, for $0 < \alpha < 1$, the t component of conserved vector is obtained as follows:

$$C^t = x\left[\left(\frac{3-\beta}{2\beta}\right)I_t^{1-\alpha}(u) - \frac{x}{\beta}I_t^{1-\alpha}(u_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tu_t)\right], \tag{80}$$

and for $1 < \alpha < 2$, the t component is calculated as below:

$$C^t = x\left[\left(\frac{3-\beta}{2\beta}\right)D_t^{\alpha-1}(u) - \frac{x}{\beta}D_t^{\alpha-1}(u_x) - \frac{1}{\alpha}D_t^{\alpha-1}(tu_t)\right]. \tag{81}$$

5 Conclusion

In this article, the symmetry approach has been extended in order to perform Lie symmetry analysis and find conservation laws for the space–time fractional nonlinear PDEs. Two illustrative examples, namely the STFGP equation and the STFgKdV equation, have been studied by using the symmetry analysis. With the aid of the new conservation theorem and the generalization of the Noether operators, the conservation laws for each of the equations are obtained successfully.

Acknowledgements Rajesh Kumar Gupta thanks the University Grants Commission for sponsoring this research under Research Award Scheme (F. 30-105/2016 (SA-II)).

References

1. Herrmann, R.: Fractional Calculus: An Introduction for Physicist. World Scientific, New Jersey (2011)
2. Herrmann, R.: Common aspects of q-deformed Lie algebras and fractional calculus. Phys. A **389**, 4613–4622 (2010)
3. Herrmann, R.: Towards a geometric interpretation of generalized fractional integrals- Erdélyi-Kober type integrals on R^n, as an example. Fract. Calc. Appl. Anal. **17**, 361–370 (2014)
4. Herrmann, R.: Infrared spectroscopy of diatomic molecules—a fractional calculus approach. Int. J. Mod. Phys. B **27**, 1350019 (2013)
5. Li, C., Zeng, F.: Numerical Methods for Fractional Calculus. CRC Press Book, Boca Raton (2015)
6. Das, S.: Functional Fractional calculus. Springer, Berlin (2011)
7. Atangana, A., Secer, S.: A note on fractional order derivatives and table of fractional derivatives of some special functions. Abst. Appl. Anal. **2013**, Article ID-279681 (2013)
8. El-Nabulsi, R.A.: The fractional white dwarf hydrodynamical nonlinear differential equation and emergence of quark stars. Appl. Math. Comput. **218**, 28–37 (2011)
9. El-Nabulsi, R.A.: Fractional functional with two occurrences of integrals and asymptotic optimal change of drift in the Black-Scholes model. Acta Math. Vietnam. **40**, 689–703 (2015)
10. El-Nabulsi, R.A.: Non-standard fractional Lagrangians. Nonlinear Dyn. **74**, 381–394 (2013)
11. El-Nabulsi, R.A.: Calculus of variations with hyperdifferential operators from Tabasaki–Takebe–Toda lattice arguments. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **107**, 419–436 (2013)
12. El-Nabulsi, R.A.: Fractional action oscillating phantom cosmology with conformal coupling. Eur. Phys. J. Plus **130**, 102–112 (2015)
13. El-Nabulsi, R.A.: Modifications at large distances from fractional and fractal arguments. Fractals **18**, 185–190 (2010)
14. El-Nabulsi, R.A.: Fractional oscillators from non-standard Lagrangians and time-dependent fractional exponent. Comput. Appl. Math. **33**, 163–179 (2014)
15. El-Nabulsi, R.A., Wu, C.G.: Fractional complexified field Theory from Saxena-Kumbhat fractional integral, fractional derivative of order (α, β) and dynamical fractional integral exponent. Afr. Diaspora J. Math. New Series **13**, 45–61 (2012)
16. Magin, R.L., Royston, T.J.: Fractional-order elastic models of cartilage: a multi-scale approach. Commun. Nonlinear Sci. Numer. Simul. **15**, 657–664 (2010)
17. Goldfain, E.: Fractional dynamics and the standard model for particle physics. Commun. Nonlinear Sci. Numer. Simul. **13**, 1397–1404 (2008)
18. Goldfain, E.: Complexity in quantum field theory and physics beyond the standard model. Chaos Solitons Fractals **28**, 913–922 (2006)
19. Goldfain, E.: Fractional dynamics, Cantorian space time and the gauge hierarchy problem. Chaos Solitons Fractals **22**, 513–520 (2004)
20. Butera, S., Di Paola, M.: A physically based connection between fractional calculus and fractal geometry. Ann. Phys. **350**, 146–158 (2014)
21. Jumarie, G.: Probability calculus of fractional order and fractional Taylor's series application to Fokker–Planck equation and information of non-random functions. Chaos Solitons Fractals **40**, 1428–1448 (2009)
22. Michelitsch, T.M., Collet, B.A., Riascos, A.P., Nowakowski, A.F., Nicolleau, F.C.G.A.: A fractional generalization of the classical lattice dynamics approach. Chaos Solitons Fractals **92**, 43–50 (2016)

23. El-Nabulsi, R.A.: Fractional variational approach with non-standard power-law degenerate Lagrangians and a generalized derivative operator. Tbil. Math. J. **9**, 279–293 (2016)
24. Silva, M.F., Machado, J.T., Lopes, A.M.: Fractional order control of a hexapod robot. Nonlinear Dyn. **38**, 417–433 (2004)
25. El-Nabulsi, R.A.: Fractional quantum Euler–Cauchy equation in the Schrödinger picture, complexified harmonic oscillators and emergence of complexified Lagrangian and Hamiltonian dynamics. Mod. Phys. Lett. B **23**, 3369–3386 (2009)
26. El-Nabulsi, R.A.: Fractional elliptic operators from a generalized Glaeske-Kilbas-Saigo-Mellin transform. Funct. Anal. Approx. Comput. **7**, 29–33 (2015)
27. El-Nabulsi, R.A.: Glaeske-Kilbas-Saigo fractional integration and fractional Dixmier trace. Acta Math. Vietnam. **37**, 149–160 (2012)
28. Zhang, Y., Papelis, C.: Particle-tracking simulation of fractional diffusion-reaction processes. Phys. Rev. E **84**, 066704 (2011)
29. Zhang, Y., Benson, D.A., Reeves, D.M.: Time and space nonlocalities underlying fractional-derivative models: distinction and literature review of field applications. Adv. Water Resour. **32**, 561–581 (2009)
30. El-Nabulsi, R.A.: Fractional variational symmetries of Lagrangians, the fractional Galilean transformation and the modified Schrödinger equation. Nonlinear Dyn. **81**, 939–948 (2015)
31. Kiryakova, V.: Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, Longman Group, Harlow (1994)
32. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
33. Fellah, Z.E.A., Depollier, C.: Application of fractional calculus to the sound waves propagation in rigid porous materials: validation via ultrasonic measurements. Acta Acust. United Acust. **88**, 34–39 (2002)
34. Kulish, V.V., Lage, J.L.: Application of fractional calculus to fluid mechanics. J. Fluids Eng. **124**, 804 (2002)
35. Douglas, J.F.: Some Applications of Fractional Calculus to Polymer Science, vol. 102. Wiley, Hoboken (1997)
36. Vinagre, B.M., Chen, Y.Q.: Fractional calculus applications in automatic control and robotics. In: 41st IEEE Conference on decision and control Tutorial Workshop 2, Las Vegas, 2002
37. Debnath, L.: Recent applications of fractional calculus to science and engineering. Int. J. Math. Math. Sci. **54**, 3413–3442 (2003)
38. El-Nabulsi, R.A.: A cosmology governed by a fractional differential equation and the generalized Kilbas-Saigo-Mittag-Leffler function. Int. J. Theor. Phys. **55**, 625–635 (2016)
39. El-Nabulsi, R.A.: The fractional Boltzmann transport equation. Comput. Math. Appl. **62**, 1568–1575 (2011)
40. El-Nabulsi, R.A.: Implications of the Ornstein-Uhlenbeck-like fractional differential equation in cosmology. Rev. Mex. Fis. **62**, 240–250 (2016)
41. Zhang, Y.: Moments for tempered fractional advection-diffusion equations. J. Stat. Phys. **139**, 915–939 (2010)
42. El-Nabulsi, R.A.: The fractional kinetic Einstein-Vlasov system and its implications in Bianchi spacetimes geometry. Int. J. Theor. Phys. **53**, 2712–2726 (2014)
43. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
44. Wang, G.W., Liu, X.Q., Zhang, Y.Y.: Lie symmetry analysis to the time fractional generalized fifth order KdV equation. Commun. Nonlinear Sci. Numer. Simul. **18**, 2321–2326 (2013)
45. Gazizov, R.K., Kasatkin, A.A., Lukashchuk, SYu.: Continuous transformation groups of fractional differential equations. Vestn. USATU **9**, 125–135 (2007)
46. Gazizov, R.K., Kasatkin, A.A., Lukashchuk, SYu.: Symmetry properties of fractional diffusion equations. Phys. Scr. **T136**, 014016 (2009)
47. Buckwar, E., Luchko, Y.: Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. J. Math. Anal. Appl. **227**, 81–97 (1998)
48. El-Kinani, E.H., Ouhadan, A.: Lie symmetry analysis of some time fractional partial differential equations. Int. J. Mod. Phys. Conf. Ser. **38**, 1560075 (2015)
49. Yasar, E., Yildirim, Y., Khalique, C.M.: Lie symmetry analysis, conservation laws and exact solutions of the seventh-order time fractional Sawada–Kotera–Ito equation. Res. Phys. **6**, 322–328 (2016)
50. Gaur, M., Singh, K.: Symmetry analysis of time fractional potential Burgers' equation. Math. Comm. **22**, 1–11 (2017)
51. Singla, K., Gupta, R.K.: On invariant analysis of some time fractional nonlinear systems of partial differential equations. I. J. Math. Phys. **57**, 101504 (2016)
52. Luchko, Yu., Gorenflo, R.: Scale-invariant solutions of a partial differential equation of fractional order. Fract. Calc. Appl. Anal. **1**, 63–78 (1998)
53. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
54. Ibragimov, N.H., Avdonina, E.D.: Nonlinear self-adjointness, conservation laws, and the construction of solutions of partial differential equations using conservation laws. Russ. Math. Surv. **68**, 889–921 (2013)
55. Lukashchuk, S.Y.: Conservation laws for time-fractional subdiffusion and diffusion-wave equations. Nonlinear Dyn. **80**, 791–802 (2015)
56. Frederico, G.S.F., Torres, D.F.M.: A formulation of Noethers theorem for fractional problems of the calculus of variations. J. Math. Anal. Appl. **334**, 834–846 (2007)
57. Atanackovic, T.M., Konjik, S., Pilipovic, S., Simic, S.: Variational problems with fractional derivatives: invariance conditions and Nöethers theorem. Nonlinear Anal. **71**, 1504–1517 (2009)
58. Ibragimov, N.H.: A new conservation theorem. J. Math. Anal. Appl. **333**, 311–328 (2007)
59. Wang, G., Kara, A.H., Fakhar, K.: Symmetry analysis and conservation laws for the class of time-fractional nonlinear dispersive equation. Nonlinear Dyn. **82**, 281–287 (2015)
60. Rui, W., Xiangzhi, Z.: Lie symmetries and conservation laws for the time fractional Derrida–Lebowitz–Speer–Spohn equation. Commun. Nonlinear Sci. Numer. Simul. **34**, 38–44 (2016)
61. Gazizov, R.K., Ibragimov, N.H., Lukashchuk, SYu.: Nonlinear self-adjointness, conservation laws and exact solutions

of time-fractional Kompaneets equations. Commun. Nonlinear Sci. Numer. Simul. **23**, 153–163 (2015)
62. Zhang, Y.: Formulation and solution to time-fractional generalized Korteweg-de Vries equation via variational methods. Adv. Differ. Equ. **2014**, 65 (2014). doi:10.1186/1687-1847-2014-65
63. El-Wakil, S.A., Abulwafa, E.M., Zahran, M.A., Mahmoud, A.A.: Time-fractional KdV equation: formulation and solution using variational methods. Nonlinear Dyn. **65**, 55–63 (2011)
64. Camassa, R., Holm, D., Hyman, J.: A new integrable shallow water equation. Adv. Appl. Mech. **31**, 1–33 (1994)
65. Johnson, R.S.: Camassa–Holm, Korteweg–de Vries and related models for waterwaves. J. Fluid Mech. **455**, 63–82 (2002)
66. Zhang, Y.: Solution of the KdV equation with fractional time derivative via variational method. Electron. J. Differ. Equ. **66**, 1–12 (2014)
67. Atangana, A., Secer, A.: The time-fractional coupled KdV equations. Abs. Appl. Anal. **2013**, Article ID-947986 (2013)
68. Arikogulo, A., Ozkol, I.: Solution of fractional differential equations by using differential transform method. Chaos Solitons Fractals **34**, 1473–1481 (2007)
69. Ertürk, V.E., Momani, S.: Solving systems of fractional differential equations using differential transform method. J. Comput. Appl. Math. **215**, 142–151 (2008)
70. Cooper, F., Khare, A., Saxena, A.: Exact elliptic compactons in generalized Korteweg–De Vries equations. Complexity **11**, 30–34 (2006)
71. Rosenau, P., Hyman, J.M.: Compactons: solitons with finite wavelength. Phys. Rev. Lett. **70**, 564–567 (1993)
72. Fan, X., Yang, S., Zhao, D.: Travelling wave solutions for the Gilson-Pickering equation by using the simplified G'/G-expansion method. Int. J. Nonlinear Sci. **8**, 368–373 (2009)
73. Gilson, C., Pickering, A.: Factorization and Painlevé analysis of a class of nonlinear third-order partial differential equations. J. Phys. A Math. Gen. **28**, 2871–2888 (1995)
74. Chen, A., Huang, W., Tang, S.: Bifurcations of travelling wave solutions for the Gilson-Pickering equation. Nonlinear Anal. Real World Appl. (2008). doi:10.1016/j.nonrwa.2008.07.005
75. Osler, T.J.: Leibniz rule for fractional derivatives generalized and an application to infinite series. SIAM J. Appl. Math. **18**, 658–674 (1970)
76. Ibragimov, N.H.: Nonlinear self-adjointness and conservation laws. J. Phys. A Math. Theor. **44**, 432002 (2011)