

Invariant solutions of Biswas-Milovic equation

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Abstract The Biswas-Milovic equation in generalized form and with power law nonlinearity is analyzed for Lie symmetries. The classical Lie group method is applied to derive symmetries of this equation, and the ordinary differential equations deduced are further studied; and some exact solutions are obtained.

Keywords Biswas-Milovic equation · Lie symmetries · Exact solutions

1 Introduction

There has been various forms of nonlinear Schrödinger's equation (NLSE) [1, 2, 8, 10] studied in various areas of nonlinear optics, plasma physics and many other areas. Biswas and Milovic [2] studied generalized form of the NLSE and obtained 1-soliton solutions of this equation. There are several forms of Schrödinger's equations that describe the soliton propagation, and many results are reported in last couple of decades [3, 11, 12, 14–17, 19–24]. The Biswas-Milovic equation (BME) [2, 15–17] that serves as a generalized version of the usual nonlinear Schrödinger's equation is in the following form:

$$iq_t^m + aq_{xx}^m + bF(|q|^2)q^m = 0 \quad (1)$$

where $q = q(x, t)$ is a complex-valued function. In Eq. (1), the first term represents the generalized evolution [6, 18] term. The second term gives the generalized version of group velocity dispersion (GVD), while the third term is the generalized form of non-Kerr law media.

For power law nonlinearity, we have $F(u) = u^n$, then Eq. (1), reduces to

$$iq_t^m + aq_{xx}^m + b|q|^{2n}q^m = 0. \quad (2)$$

Eslami and [6] used several integration schemes to derive solitons and other solutions to the model (1) for various forms of $F(|q|^2)$. Zhou et al. [18] used extended trail equation method to obtain soliton solutions such as bright, dark and singular solitons of Biswas-Milovic equation for various forms of $F(|q|^2)$. So authors [2, 6, 18] have not obtained solutions for general form of Biswas-Milovic equations (1).

In this paper, we will study BME (1) and (2) by the Lie classical method. Firstly, Lie classical method will be used to obtain symmetries of BME (1) and (2). Further using symmetries BME will be reduced to ordinary differential equations and corresponding to the reduction, exact solutions of the Biswas-Milovic equations (BME) (1) and (2) will be obtained.

2 Symmetry analysis

In this section, we will perform Lie symmetry analysis [4, 5, 7–9, 13] for the Biswas-Milovic equations (1) and (2). Firstly, let us assume

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$$q(x, t) = u(x, t)e^{iv(x,t)}, \tag{3}$$

where u will depend upon the form of nonlinear function F in Eq. (1).

2.1 Lie symmetry analysis for equation (1)

Substituting (3) into (1), and separating real and imaginary parts, we have

$$\begin{aligned} muu_t + 2am^2uu_xv_x + am^2v_{xx} &= 0 \\ -mv_tu^2 + am^2u_x^2 - am^2v_x^2u^2 \\ + amuu_{xx} - am^2u_x^2 + bF(u^2)u^2 &= 0. \end{aligned} \tag{4}$$

Let us consider the Lie group of point transformations

$$\begin{aligned} t^* &= t + \epsilon\tau(x, t, u, v) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi(x, t, u, v) + O(\epsilon^2) \\ u^* &= u + \epsilon\eta(x, t, u, v) + O(\epsilon^2) \\ v^* &= v + \epsilon\phi(x, t, u, v) + O(\epsilon^2), \end{aligned} \tag{5}$$

which leaves the system of Eq. (4) invariant. The infinitesimal transformations (5) are such that if u, v is solution of Eq. (4), then u^*, v^* is also a solution. The method for determining the similarity variables of (4) mainly consists of finding the infinitesimals τ, ξ, ϕ and η , which are functions of x, t, u, v .

The geometric field of nonlinear evolution equations is

$$\begin{aligned} V &= \xi(x, t, u, v)\partial_x + \tau(x, t, u, v)\partial_t \\ &+ \eta(x, t, u, v)\partial_u + \phi(x, t, u, v)\partial_v, \end{aligned} \tag{6}$$

where τ, ξ, ϕ and η , are coefficient functions of vector field to be determined.

If vector field (6) generates a symmetry of system of Eq. (4), then vector field V must satisfy the symmetry condition

$$pr^{(2)}V(\Delta)|_{\Delta=0} = 0, \tag{7}$$

where Δ is system of Eq. (4).

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial x}, V_3 = \frac{\partial}{\partial v}, \\ V_4 &= t\frac{\partial}{\partial x} + \frac{x}{2am}\frac{\partial}{\partial v}. \end{aligned} \tag{8}$$

2.2 Lie symmetry analysis for equation (2)

Substituting (3) into (2), and separating real and imaginary parts, we have

$$\begin{aligned} muu_t + 2am^2uu_xv_x + am^2v_{xx} &= 0 \\ -mv_tu^2 + am^2u_x^2 - am^2v_x^2u^2 \\ + amuu_{xx} - am^2u_x^2 + bu^{2n+2} &= 0. \end{aligned} \tag{9}$$

Using the classical Lie symmetry method as mentioned above, we obtain following vector fields for system of Eq. (4):

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = \frac{\partial}{\partial v}, \\ X_4 &= \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{2n}\frac{\partial}{\partial u} \\ X_5 &= t\frac{\partial}{\partial x} + \frac{x}{2am}\frac{\partial}{\partial v}. \end{aligned} \tag{10}$$

3 Similarity reductions and exact solutions

In preceding section, we obtain the vector fields of the Eqs. (1) and (2). Now, we will deal with the similarity reductions and exact solutions to the Biswas-Milovic equations.

3.1 Similarity reductions and exact solutions of equation (1)

For similarity reduction of Eq. (1), we consider two cases:

- (i) V_4 (ii) $V_1 + \lambda V_2 + \mu V_3$ (iii) $V_2 + \mu V_3$

For vector field V_4

Similarity variables are

$$\begin{aligned} \xi &= t \\ u &= G(\xi) \\ v &= \frac{1}{t}\left(\frac{x^2}{4am} + H(\xi)\right), \end{aligned} \tag{11}$$

where ξ is new independent variable and G, H are new dependent variables.

Using (11), in system of Eq. (4), we obtain

$$\begin{aligned} 2m\xi G' + G &= 0 \\ H + \xi H' + b\xi^2 F(G^2) &= 0, \end{aligned} \tag{12}$$

where ' denotes derivative with respect to ξ .

We obtain following solution of system of ordinary differential Eq. (12)

$$\begin{aligned} G &= C_1\xi^{-\frac{1}{2m}} \\ H &= \left(\frac{b}{m}\int F(G^2)d\xi + C_2\right)\xi, \end{aligned} \tag{13}$$

where C_1, C_2 are arbitrary constants.

Corresponding solution of system of Eq. (2) is

$$u = C_1 t^{-\frac{1}{2m}}$$

$$v = \frac{x^2}{4amt} + \left(\frac{b}{m} \int F(u^2) dt + C_2 \right) t \tag{14}$$

where C_1 is arbitrary constant.

So, corresponding solution of main Eq. (1) is

$$q(x, t) = C_1 t^{-\frac{1}{2m}} e^i \left(\frac{x^2}{4amt} + \left(\frac{b}{m} \int F(C_1^2 t^{-\frac{1}{m}}) dt + C_2 \right) t \right) \tag{15}$$

For vector field $V_1 + \lambda V_2 + \mu V_3$

Corresponding similarity variables are

$$\xi = x - \lambda t$$

$$u = G(\xi)$$

$$v = \mu t + H(\xi), \tag{16}$$

where ξ is new independent variable and G, H are new dependent variables.

Using (16) into system of Eq. (4), we have

$$aH''G + 2amH'G' - \lambda G' = 0$$

$$-m\mu G^2 + m\lambda G^2 H' + am^2 G'^2 - am^2 H'^2 G^2 + amGG'' - amG'^2 + bF(G^2)G^2 = 0, \tag{17}$$

where ' denotes derivative with respect to ξ .

From first equation of system of ordinary differential Eq. (17), we have

$$H = \frac{1}{2am} \int \left(\lambda + \frac{C_1}{G^{2m}} \right) d\xi + C_2, \tag{18}$$

where C_1, C_2 are arbitrary constants.

$$4a^2(G^m)'' + (-4am\mu + \lambda^2)G^m - C_1^2 G^{-3m} + 4abF(G^2)G^m = 0. \tag{19}$$

So solution of main Eq. (1) is given by

$$q(x, t) = G e^{i(\mu t + H)}, \tag{20}$$

where H is given by (18) and G satisfies the Eq. (19).

For vector field $V_2 + \mu V_3$

Corresponding similarity variables are

$$\xi = t$$

$$u = G(\xi)$$

$$v = \mu x + H(\xi), \tag{21}$$

where ξ is new independent variable and H, G are new dependent variables.

Using (21) in system of Eq. (4), we have

$$mGG' = 0$$

$$mH' + am^2\mu^2 - bF(G^2) = 0, \tag{22}$$

where ' denotes derivative with respect to ξ .

We obtain following solution of (22)

$$G = C_1$$

$$H = \left(\frac{b}{m} F(G^2) - am\mu^2 \right) t + C_2, \tag{23}$$

where C_1, C_2 are arbitrary constants.

Corresponding solution of main Eq. (1) is

$$q(x, t) = C_1 e^{i\left(\mu x + \left(\frac{b}{m} F(C_1^2) - am\mu^2\right)t + C_2\right)}. \tag{24}$$

3.2 Similarity reductions and exact solutions of equation (2)

For similarity reduction of Eq. (2), we consider following vector fields (i) V_4

For vector field X_4

For vector field V_4 , similarity variables are

$$\sigma = \frac{x^2}{t}$$

$$u = x^{-\frac{1}{n}} P(\sigma)$$

$$v = Q(\sigma), \tag{25}$$

where σ is new independent variable and P, Q are new dependent variables.

Using (25) in (9), we have

$$n\sigma P' - 8am n\sigma P' Q' + 4am P Q' - 4an\sigma P Q'' - 2an P Q' = 0$$

$$-m n^2 \sigma^2 P^2 Q' - 4am^2 n^2 \sigma^2 P'^2 + 4am^2 n\sigma P' P - am^2 P^2 + 4am^2 n^2 \sigma^2 P^2 Q'^2$$

$$- 4am n^2 \sigma^2 P P'' - 2am n^2 \sigma P P' - am n P^2 + 4am n^2 \sigma^2 P'^2 - b n^2 P^{2n+2} = 0, \tag{26}$$

where ' denotes derivative with respect to σ .

For $n = m$, we obtain following solution of system (26)

$$P = \left(-\frac{1}{18ab} \right)^{\frac{1}{2m}} \sigma^{\frac{1}{m}}$$

$$Q = \frac{\sigma}{6am}. \tag{27}$$

Corresponding solution of main Eq. (2) is given by

$$q(x, t) = \left(-\frac{1}{18ab}\right)^{\frac{1}{2m}} \left(\frac{x}{t}\right)^{\frac{1}{m}} e^{i\left(\frac{x^2}{6amt}\right)}, \tag{28}$$

with $n = m$.

For vector field X_5

Similarity variables are

$$\begin{aligned} \sigma &= t \\ u &= P(\sigma) \\ v &= \frac{1}{t} \left(\frac{x^2}{4am} + Q(\sigma)\right), \end{aligned} \tag{29}$$

where σ is new independent variable and G, H are new dependent variables.

Using (11), in system of Eq. (4), we obtain

$$\begin{aligned} 2m\sigma P' + P &= 0 \\ Q + \sigma Q' + b\sigma^2 P^{2n} &= 0, \end{aligned} \tag{30}$$

where ' denotes derivative with respect to σ .

We obtain following solution of system (30)

$$\begin{aligned} P &= C_1 \sigma^{-\frac{1}{2m}} \\ Q &= \frac{bC_1^{2n}}{m-n} \sigma^{-\frac{n}{m}+2} + C_2 \sigma, \end{aligned} \tag{31}$$

where C_1, C_2 are arbitrary constants.

Using (29) and (13), solution of main Eq. (2) is given by

$$q(x, t) = C_1 t^{-\frac{1}{2m}} e^{i\left(\frac{x^2}{4amt} + \frac{bC_1^{2n}}{m-n} t^{-\frac{n}{m}+1} + C_2\right)}. \tag{32}$$

For vector field $X_1 + \lambda X_2 + \mu X_3$

Corresponding similarity variables are

$$\begin{aligned} \sigma &= x - \lambda t \\ u &= P(\sigma) \\ v &= \mu t + Q(\sigma), \end{aligned} \tag{33}$$

where σ is new independent variable and P, Q are new dependent variables.

Using (33) into system of Eq. (9), we have

$$\begin{aligned} aQ''P + 2amQ'P' - \lambda P' &= 0 \\ -m\mu P^2 + m\lambda P^2 Q' + am^2 P'^2 - am^2 Q'^2 P^2 \\ + am P P'' - am P'^2 + b P^{2n+2} &= 0, \end{aligned} \tag{34}$$

where ' denotes derivative with respect to σ .

From first equation of system of ordinary differential Eq. (34), we have

$$Q = \frac{1}{2am} \int \left(\lambda + \frac{C_1}{P^{2m}}\right) d\sigma + C_2, \tag{35}$$

where C_1, C_2 are arbitrary constants.

$$\begin{aligned} 4a^2(P^m)'' + (-4am\mu + \lambda^2) P^m - C_1^2 P^{-3m} \\ + 4ab P^{2n+m} &= 0. \end{aligned} \tag{36}$$

So solution of main Eq. (2) is given by

$$q(x, t) = P e^{i(\mu t + Q)}, \tag{37}$$

where Q is given by (35) and P satisfies the Eq. (36).

Solving (37) for $C_1=0$ and $\lambda^2=4am\mu$, we obtain following solution

$$P(\sigma) = \left(-\frac{a}{b} \left(\frac{m^2}{n^2} + \frac{m}{n}\right)\right)^{\frac{1}{2n}} \sigma^{-\frac{1}{n}}. \tag{38}$$

From (35), we have

$$Q(\sigma) = \frac{\lambda}{2am} \sigma + C_2. \tag{39}$$

Corresponding solution of main Eq. (1) is

$$\begin{aligned} q(x, t) = \left(-\frac{a}{b} \left(\frac{m^2}{n^2} + \frac{m}{n}\right)\right)^{\frac{1}{2n}} (x \\ - 2\sqrt{am\mu}t)^{-\frac{1}{n}} e^{i\left(\sqrt{\frac{\mu}{am}}(x-2\sqrt{am\mu}t) + C_2\right)}. \end{aligned} \tag{40}$$

Now, let us consider that $\lambda^2 \neq 4am\mu$ and $C_1=0$. In this case, we assume that

$$P(\sigma) = \frac{A}{\cosh^p(B\sigma)}, \tag{41}$$

where p, A and B will be determined.

Substituting (41) and (39) with $C_1=0$ in (36), we obtain

$$p = \frac{1}{n}, \tag{42}$$

$$B = \left[\frac{bn^2 A^{2n}}{am(m+n)}\right]^{\frac{1}{2}} \tag{43}$$

and

$$\mu = \frac{\lambda^2 n^2 + 4a^2 m^2 B^2}{4amn^2}. \tag{44}$$

Corresponding solution of Eq. (2) is given by

$$q(x, t) = \frac{A}{\cosh^{\frac{1}{n}} B(x - \lambda t)} e^{i\left(\mu t + \frac{\lambda}{2am}(x - \lambda t) + C_2\right)} \tag{45}$$

with conditions (43)–(44).

Remark 1 Solution (32) can be obtained from (15) by just taking $F(u^2) = u^{2n}$.

4 Conclusions

This paper analyzed and studied the Biswas-Milovic equation using Lie classical method. Firstly, Lie classical method is performed on Biswas-Milovic equations (1) and using similarity variables equation is reduced to ordinary differential equations. Corresponding to reductions, some exact solutions of Biswas-Milovic equations (1) are obtained. Then, Biswas-Milovic equation for power law nonlinearity (2) is also studied, and some exact solutions are obtained. The results of this paper will be helpful toward further research in this direction. In future work, BME will be studied for variable coefficients.

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