



ON KROPINA-RANDERS CHANGE OF m th-ROOT FINSLER METRIC

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Abstract

In the present paper, we consider the Kropina-Randers change of m th root Finsler metric. Firstly, we find the fundamental metric tensors of the Kropina-Randers transformed m th root Finsler metric, and then the necessary and sufficient condition under which the Kropina-Randers change of the m th root Finsler metric is locally dually flat. Further, we prove that the Kropina-Randers change of m th root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

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1. Introduction

Let M be an n -dimensional C^∞ manifold, TM its tangent bundle. Let $F = A^{\frac{1}{m}}$ be a Finsler metric on M , where A is given by $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 i_2 \dots i_m}$ symmetric in all its indices ([3], [10-12], [14]). Then F is called an m th-root Finsler metric. Suppose that A_{ij} is a positive definite tensor and A^{ij} denotes its inverse. For an m th root metric F , put

$$A_i = \frac{\partial A}{\partial y^i}, A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, A_{x^i} = \frac{\partial A}{\partial x^i}, A_0 = A_{x^i} y^i. \quad (1)$$

Then the following hold:

$$l_i = \frac{1}{m} A_i F^{1-m}, \quad (2)$$

$$g_{ij} = \frac{A^{\frac{2-2m}{m}}}{m^2} [m A A_{ij} + (2-m) A_i A_j], \quad (3)$$

$$y^i A_i = mA, y^i A_{ij} = (m-1) A_j, y_i = \frac{1}{m} A^{\frac{2-m}{m}} A_i, \quad (4)$$

$$A^{ij} A_{jk} = \delta_k^i, A^{ij} A_i = \frac{1}{m-1} y^j, A_i A_j A^{ij} = \frac{m}{m-1} A. \quad (5)$$

Let (M, F) be a Finsler manifold. For a 1-form $\beta(x, y) = b_i(x) y^i$ on M , we have a change of Finsler metric, defined by

$$F(x, y) \rightarrow \bar{F}(x, y) = f(F, \beta), \quad (6)$$

where $f(F, \beta)$ is a positively homogenous function of F . This is called a β -change of a metric. It is easy to see that, if $\|\beta\|_F = \sup_{F(x, y)=1} |b_i(x) y^i| < 1$, then \bar{F} is again a Finsler metric [9].

In this paper, we consider a special case of β -change, namely,

$$\bar{F}(x, y) = \frac{F^2(x, y)}{\beta} + \beta, \tag{7}$$

which we call the *Kropina-Randers change*. The Randers and Kropina metrics are closely related to physical theories. These metrics were introduced by Berwald in connection with a two dimensional Finsler space with rectilinear extremal and were investigated by Kropina and Randers, respectively ([7], [5]).

The main purpose of the current paper is to investigate the Kropina-Randers change of an m th-root Finsler metric $\bar{F} = \frac{A^{\frac{2}{m}}}{\beta} + \beta$. The paper is organised as follows:

First we find the fundamental metric tensors and its inverse for Kropina-Randers change of a Finsler space with m th-root metric (see Propositions 2.1 and 2.2). Next we prove that the Kropina-Randers change of an m th root Finsler metric is locally projectively flat if and only if it is locally Minkowskian (see Proposition 3.1 and Theorem 3.2). Finally, we find the necessary and sufficient condition under which the Kropina-Randers change of an m th-root Finsler metric to be locally dually flat.

2. Fundamental Metric Tensor of Kropina-Randers Change of m th-root Finsler Metric

We consider the Kropina-Randers change of the m th-root metric $F = A^{\frac{1}{m}}$ given by

$$\bar{F} = \frac{F^2}{\beta} + \beta. \tag{8}$$

The differentiation of (8) with respect to y^i yields the normalized supporting element l_i given by

$$\bar{l}_i = \frac{2l_i\beta F - b_i F^2}{\beta^2} + b_i.$$

In view of (2), we have

$$\bar{l}_i = \frac{2A_i F^{2-m}}{m\beta} - \frac{b_i F^2}{\beta^2} + b_i. \quad (9)$$

Differentiation of (9) with respect to y^j yields

$$\begin{aligned} \bar{l}_{ij} = \frac{2}{m^2\beta^3} [m\beta^2 F^{-m+2} A_{ij} - (m-2)\beta^2 F^{-2m+2} A_i A_j \\ - m\beta F^{-m+2} (A_i b_j + A_j b_i) + mF^2 b_i b_j]. \end{aligned} \quad (10)$$

From (9) and (10), we obtain

$$\begin{aligned} \bar{h}_{ij} &= \bar{F} \bar{l}_{ij} \\ &= p_1 A_{ij} + p_2 A_i A_j + p_3 (A_j b_i + A_i b_j) + p_4 b_i b_j, \end{aligned} \quad (11)$$

where

$$\begin{aligned} p_1 &= \frac{2}{m^2\beta^2} (F^{-m+4} + \beta^2 F^{-m+2}), \\ p_2 &= \frac{-2(m-2)}{m^2\beta^2} (F^{-2m+4} + F^{-2m+2}\beta^2), \\ p_3 &= \frac{-2}{m\beta^3} (F^{-m+4} + F^{-m+2}\beta^2), \\ p_4 &= \frac{2}{m\beta^4} (F^4 + m\beta^2 F^2). \end{aligned}$$

From (9) and (11), the fundamental metric tensor \bar{g}_{ij} of Finsler space (M, \bar{F}) is given by

$$\begin{aligned} \bar{g}_{ij} &= \bar{h}_{ij} + \bar{l}_i \bar{l}_j \\ &= \rho_0 A_{ij} + \rho_1 (b_i A_j + b_j A_i) + \rho_2 b_i b_j + \rho_3 (A_i A_j), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \rho_0 &= \frac{2F^{-m+4}(1 + \beta^2 F^{-1})}{m\beta^3}, \\ \rho_1 &= \frac{-4F^{-2}}{m\beta}, \\ \rho_2 &= \frac{2}{m\beta^4} (F^4 + m\beta^2 F^2) + \frac{(\beta^2 - F^2)}{\beta^2}, \\ \rho_3 &= \frac{-2F^{-2m+2}}{m^2\beta^2} [(m - 4)F^2 + \beta^2]. \end{aligned} \tag{13}$$

Proposition 2.1. *For the Kropina-Randers transformed m th-root Finsler metric \bar{F} , the fundamental metric tensor \bar{g}_{ij} is given by equations (12) and (13).*

The contravariant metric tensor \bar{g}^{ij} of Finsler space (M, \bar{F}) is given by

$$\begin{aligned} \bar{g}^{ij} &= \frac{1}{\rho_0} A^{ij} - \mu \frac{b^i b^j}{\rho_0^2} - \frac{\bar{\delta}}{1 + \bar{\delta}l} \left\{ \frac{1}{\rho_0} (b^i + \lambda y^i) - \bar{\delta} b^i (b^2 + \lambda\beta) \right\} \\ &\times \left\{ \frac{1}{\rho_0} (b^j + \lambda y^j) - \bar{\delta} b^j (b^2 + \lambda\beta) \right\}, \end{aligned} \tag{14}$$

where

$$\mu = \frac{4F^{12} + 4mF^{10}\beta^2 + 4mF^8\beta^4 + m^2F^8\beta^4 + 2m^2F^6\beta^6 + m^2\beta^8F^4 - 16\beta^6}{m\beta^4F^4(2F^4 + F^2m\beta^2 + m\beta^4)},$$

$$\begin{aligned} \bar{\delta} = & [2(4F^{12} + 4mF^{10}\beta^2 + 4mF^8\beta^4 + m^2F^8\beta^4 + 2m^2F^6\beta^6 + m^2\beta^8F^4 \\ & - 16\beta^6)(F^2 + \beta^2)]/[m\beta^2F^{2-m}(4\beta^2F^{12-m} + 4\beta^4F^{10-m} \\ & + 2m\beta^4F^{10-m} + 4m\beta^6F^{8-m} + 2m\beta^8F^{6-m} + 4b^2F^{12} \\ & + 4mb^2F^{10}\beta^2 + 4mb^2F^8\beta^4 + m^2b^2\beta^4F^8 + 2m^2b^2\beta^6F^6 \\ & + b^2m^2\beta^8F^4 - 16b^2\beta^6)], \end{aligned}$$

$$l = \frac{1}{\rho_0} [b^2 + \lambda(2\beta + \lambda mF^m)] - \bar{\delta}[b^4 + \lambda(2\beta + \lambda mF^m)b_i],$$

in which

$$\rho_0 = \frac{2F^{-m+4}(1 + \beta^2F^{-1})}{m\beta^3}, \quad \lambda = \frac{F^3[(m-4)F^2 + \beta^2]}{2m(m-1)\beta}. \quad (15)$$

Proposition 2.2. For the Kropina-Randers transformed m th-root Finsler metric \bar{F} , the contravariant metric tensor \bar{g}^{ij} is given by equations (14) and (15).

3. Locally Projectively Flat Metric

A Finsler metric $F(x, y)$ on an open domain $U \subset R^n$ is said to be *locally projectively flat metric* if its geodesic coefficients G^i are in the following form:

$$G^i(x, y) = P(x, y)y^i, \quad (16)$$

where $P : TU = U \times R^n \rightarrow R$ is positively homogenous with degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$. We call $P(x, y)$ the *projective factor* of F .

Definition 3.1. A Finsler metric $F = F(x, y)$ on an open subset $U \subset R^n$ is *projectively flat* [4] if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \quad (17)$$

Now, we have the following lemma.

Lemma 3.1. *Suppose that the equation $\Phi A^2 + \Psi A + \Theta = 0$ holds, where Φ, Ψ, Θ are polynomials in y and $m > 2$. Then $\Phi = \Psi = \Theta = 0$.*

Proposition 3.1. *Let $F = A^{\frac{1}{m}}$ be an m th-root Finsler metric on an open subset $U \subset R^n$ ($n \geq 3$), where A is irreducible. Suppose that $\bar{F} = \frac{F^2}{\beta} + \beta$ is a Kropina-Randers change of F , where $\beta = b_i(x)y^i$. If \bar{F} is projectively flat metric, then it reduces to a Berwald metric.*

Proof. Let $\bar{F} = \frac{F^2}{\beta} + \beta$ be projectively flat metric. We have

$$\begin{aligned} \bar{F}_{x^k} &= \frac{2}{m\beta} A^{\frac{2-m}{m}} A_{x^k} - \frac{1}{\beta^2} A^{\frac{2}{m}} \beta_{x^k} + \beta_{x^k}, \\ \bar{F}_{x^k y^l y^k} &= \frac{1}{\beta} \left[\frac{2}{m} A^{\frac{2-m}{m}} A_{0l} + \frac{4-2m}{m} A^{\frac{2-2m}{m}} A_0 A_l \right] \\ &\quad - \frac{1}{\beta^2} \left[\frac{2}{m} A^{\frac{2-m}{m}} A_l \beta_0 + \frac{2}{m} A^{\frac{2-m}{m}} A_0 \beta_l + A^{\frac{2}{m}} \beta_{0l} - \beta^2 \beta_{0l} \right] \\ &\quad + \frac{2}{\beta^3} [A^{\frac{2}{m}} \beta_0 \beta_l]. \end{aligned}$$

From (17), we get

$$\begin{aligned} \frac{2}{m\beta} A^{\frac{2-m}{m}} A_{x^l} - \frac{1}{\beta^2} A^{\frac{2}{m}} \beta_{x^l} + \beta_{x^l} \\ - \frac{1}{\beta} \left[\frac{2}{m} A^{\frac{2-m}{m}} A_{0l} + \frac{4-2m}{m} A^{\frac{2-2m}{m}} A_0 A_l \right] \end{aligned}$$

$$+ \frac{1}{\beta^2} \left[\frac{2}{m} A^{\frac{2-m}{m}} A_l \beta_0 + \frac{2}{m} A^{\frac{2-m}{m}} A_0 \beta_l + A^{\frac{2}{m}} \beta_{0l} - \beta^2 \beta_{0l} \right]$$

$$- \frac{2}{\beta^3} [A^{\frac{2}{m}} \beta_0 \beta_l] = 0$$

which implies

$$\frac{A^{\frac{2-2m}{m}}}{\beta^3} \left[\frac{2}{m} \beta^2 \left[\left(\frac{2-m}{m} \right) A_0 A_l + A A_{0l} - A A_{x'l} \right] - \frac{2}{m} A \beta [A_l \beta_0 + A_0 \beta_l] \right.$$

$$\left. + A^2 [2\beta_0 \beta_l + \beta \beta_{x'l} - \beta \beta_{0l}] \right] = 0.$$

Multiplying by A^2 , we get

$$\frac{A^{\frac{2}{m}}}{\beta^3} \left[\frac{2}{m} \beta^2 \left[\left(\frac{2-m}{m} \right) A_0 A_l + A A_{0l} - A A_{x'l} \right] - \frac{2}{m} A \beta [A_l \beta_0 + A_0 \beta_l] \right.$$

$$\left. + A^2 [2\beta_0 \beta_l + \beta \beta_{x'l} - \beta \beta_{0l}] + A^{\frac{2m-2}{m}} \beta^3 \beta_0 \beta_l \right] = 0$$

which implies

$$\left[\frac{2}{m} \beta^2 \left[\left(\frac{2-m}{m} \right) A_0 A_l + A A_{0l} - A A_{x'l} \right] - \frac{2}{m} A \beta [A_l \beta_0 + A_0 \beta_l] \right.$$

$$\left. + A^2 [2\beta_0 \beta_l + \beta \beta_{x'l} - \beta \beta_{0l}] + A^{\frac{2m-2}{m}} \beta^3 \beta_0 \beta_l \right] = 0.$$

By Lemma 3.1, we have

$$mA(A_{0l} - A_{x'l}) - (m-2)A_0 A_l = 0. \quad (18)$$

Then, irreducibility of A and $\deg(A_l) = m - 1 < \deg(A)$ implies that A_0 is divisible by A . This means that, there is a 1-form $\theta = \theta_i y^i$ on U such that the following holds:

$$A_0 = 2mA\theta. \tag{19}$$

Then $G^i = Py^i$, where $P = \theta$. Then F is a Berwald metric. □

Theorem 3.2. *Let $F = A^{\frac{1}{m}}$ be an m th-root Finsler metric on an open subset $U \subset R^n$ ($n \geq 3$), where A is irreducible. Suppose that $\bar{F} = \frac{F^2}{\beta} + \beta$ is a Kropina-Randers change of F , where $\beta = b_i(x)y^i$. Then \bar{F} is locally projectively flat if and only if it is locally Minkowskian.*

Proof. By Proposition 3.1, if F is projectively flat, then it reduces to a Berwald metric.

Now, if $n \neq 3$, then by Numata’s theorem, every Berwald metric of non-zero scalar flag curvature \mathbf{K} must be Riemannian. This contradicts with our assumption. Then $\mathbf{K} = 0$, and in this case F reduces to a locally Minkowskian metric. □

4. Locally Dually Flat Metric

The notion of dually flat Riemannian metrics was introduced by Amari and Nagaoka [1, 2] when they studied the information geometry on Riemannian manifolds. In Finsler geometry, Shen extended the notion of locally dually flatness for Finsler metrics. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which plays a very important role in studying flat Finsler information structure. Information geometry has been emerged from investigating the geometrical structure of the family of probability distributions.

Definition 4.1. A Finsler metric $F = F(x, y)$ on a manifold M^n is said to be *locally dually flat*, if at any point there is a standard coordinate system (x^i, y^i) in TM such that

$$[F^2]_{x^k y^l} y^k = 2[F^2]_{x^l}. \quad (20)$$

Theorem 4.1. Let $F = A^{\frac{1}{m}}$ be an m th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = \frac{F^2}{\beta} + \beta$ is a Kropina-Randers change of F , where $\beta = b_i(x) y^i$. Then F is locally dually flat if and only if there exists a 1-form $\theta = \theta_i(x) y^i$ on U such that the following hold:

$$2\beta_{x^l} = -2\theta\beta_l + \beta\theta_l,$$

$$A_{x^l}\beta = -A_l\beta_0 - A_0\beta_l,$$

$$4\beta\beta_{x^l} = -\beta\beta_{0l} + 2\beta_0\beta_l,$$

where

$$\beta_{0l} = \beta_{x^k y^l} y^k, \beta_{x^l} = (b_i)_{x^l} y^i, \beta_0 = \beta_{x^i} y^i \text{ and } \beta_{0l} = (b_l)_0.$$

Proof. Let $\bar{F} = \frac{F^2}{\beta} + \beta$ be a locally dually flat Finsler metric. We have

$$\bar{F}^2 = \frac{A^{\frac{4}{m}}}{\beta^2} + 2A^{\frac{2}{m}} + \beta^2,$$

$$[\bar{F}]_{x^k}^2 = \frac{4A^{\frac{4-m}{m}} A_{x^k}}{m\beta^2} - \frac{2A^{\frac{4}{m}}\beta_{x^k}}{\beta^3} + \frac{4}{m} A^{\frac{2-m}{m}} A_{x^k} + 2\beta\beta_{x^k},$$

$$\begin{aligned}
[\bar{F}]_{x^k y^l}^2 y^k &= \frac{1}{\beta^2} \left[\frac{4}{m} A^{\frac{4-m}{m}} A_{0l} + \frac{4}{m} \frac{(4-m)}{m} A^{\frac{4-2m}{m}} A_0 A_l \right. \\
&\quad \left. + \frac{4}{m} A^{\frac{2-m}{m}} \beta^2 A_{0l} + \frac{4}{m} \frac{(2-m)}{m} \beta^2 A^{\frac{2-2m}{m}} A_0 A_l \right] \\
&\quad - \frac{2}{\beta^3} \left[\frac{4}{m} A^{\frac{4-m}{m}} A_l \beta_0 + \frac{4}{m} A^{\frac{4-m}{m}} A_0 \beta_l + A^{\frac{4}{m}} \beta_{0l} - \beta^4 \beta_{0l} \right] \\
&\quad + \left[\frac{6}{\beta^4} A^{\frac{4}{m}} \beta_0 \beta_l + 2\beta_0 \beta_l \right].
\end{aligned}$$

From (20), we get

$$\begin{aligned}
&2 \left[\frac{4A^{\frac{4-m}{m}} A_{x^l}}{m\beta^2} - \frac{2A^{\frac{4}{m}} \beta_{x^l}}{\beta^3} + \frac{4}{m} A^{\frac{2-m}{m}} A_{x^l} + 2\beta \beta_{x^l} \right] \\
&= \frac{1}{\beta^2} \left[\frac{4}{m} A^{\frac{4-m}{m}} A_{0l} + \frac{4}{m} \frac{(4-m)}{m} A^{\frac{4-2m}{m}} A_0 A_l + \frac{4}{m} A^{\frac{2-m}{m}} \beta^2 A_{0l} \right. \\
&\quad \left. + \frac{4}{m} \frac{(2-m)}{m} \beta^2 A^{\frac{2-2m}{m}} A_0 A_l \right] \\
&\quad - \frac{2}{\beta^3} \left[\frac{4}{m} A^{\frac{4-m}{m}} A_l \beta_0 + \frac{4}{m} A^{\frac{4-m}{m}} A_0 \beta_l + A^{\frac{4}{m}} \beta_{0l} - \beta^4 \beta_{0l} \right] \\
&\quad + \left[\frac{6}{\beta^4} A^{\frac{4}{m}} \beta_0 \beta_l + 2\beta_0 \beta_l \right]. \tag{21}
\end{aligned}$$

Multiplying (21) by $A^{\frac{-2+2m}{m}}$, we get

$$\begin{aligned}
& A^{\frac{(2+m)}{m}} \left[\frac{8A_{x^l}}{m\beta^2} - \frac{4}{m\beta^2} A_{0l} + \frac{4}{m\beta^3} A_l\beta_0 + \frac{4}{m\beta^3} A_0\beta_l \right] \\
& + A^{\frac{2}{m}} \left[\frac{4(4-m)}{m^2\beta^2} A_0A_l \right] + A \left[\frac{8}{m} A_{x^l} - \frac{4}{m} A_{0l} \right] \\
& + \frac{4}{m} \frac{(2-m)}{m} A_0A_l - A^{\frac{(2+2m)}{m}} \left[\frac{2\beta_{x^l}}{\beta^3} - \frac{2}{\beta^3} \beta_{0l} + \frac{6}{\beta^4} \beta_0\beta_l \right] \\
& + A^{\frac{(2-2m)}{m}} [4\beta_{x^l} + \beta\beta_{0l} - 2\beta_0\beta_l] = 0. \tag{22}
\end{aligned}$$

Simplifying (22), we get

$$\begin{aligned}
& \frac{A^{\frac{2}{m}}}{\beta^4} \left[A^2(2\beta\beta_{x^l} - 2\beta\beta_{0l} + 6\beta_0\beta_l) \right. \\
& \left. + A \left(\frac{8A_{x^l}\beta^2}{m} - \frac{4\beta^2}{m} A_{0l} + \frac{4\beta}{m} A_l\beta_0 + \frac{4\beta}{m} A_0\beta_l \right) \right. \\
& \left. + \left(\frac{(8-4m)}{m^2} \beta^4 A_0A_l \right) \right] \\
& + A \left(\frac{8}{m} A_{x^l} - \frac{4}{m} A_{0l} \right) + \left(\frac{4}{m} \frac{(2-m)}{m} A_0A_l \right) \\
& + A^{\frac{(2-2m)}{m}} [4\beta_{x^l} + \beta\beta_{0l} - 2\beta_0\beta_l] = 0. \tag{23}
\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}
 \beta\beta_{x^l} &= \beta\beta_{0l} - 3\beta_0\beta_l, \\
 4A_{x^l}\beta &= \beta A_{0l} - A_l\beta_0 - A_0\beta_l, \\
 2A_{x^l} &= A_{0l}, \\
 4\beta\beta_{x^l} &= -\beta\beta_{0l} + 2\beta_0\beta_l
 \end{aligned} \tag{24}$$

which implies

$$\begin{aligned}
 \beta\beta_{x^l} &= \beta\beta_{0l} - 3\beta_0\beta_l, \\
 A_{x^l}\beta &= -A_l\beta_0 - A_0\beta_l, \\
 4\beta\beta_{x^l} &= -\beta\beta_{0l} + 2\beta_0\beta_l.
 \end{aligned} \tag{25}$$

From (25), we have

$$\beta(\beta_{x^l} - \beta_{0l}) = -3\beta_0\beta_l. \tag{26}$$

Equation (26) implies that there exists a 1-form $\theta = \theta_i(x)y^i$ on U such that

$$\beta_0 = \theta\beta. \tag{27}$$

From (27), we get

$$\beta_{0l} = \theta\beta_l + \beta\theta_l - \beta_{x^l}. \tag{28}$$

Substituting (27) and (28) into (26) yields

$$2\beta_{x^l} = -2\theta\beta_l + \beta\theta_l. \tag{29}$$

The converse is a direct computation. This completes the proof. \square

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