Four-dimensional Conformally Flat Berwald and Landsberg Spaces

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Abstract

The problem of conformal transformation and conformal flatness of Finsler spaces has been studied by so many researchers [6, 16, 17, 20, 21] . Recently, Prasad et. al [19] have studied three dimensional conformally flat Landsberg and Berwald spaces and have given some important results. The purpose of the present paper is to extend the idea of conformal change to four dimensional Finsler spaces and find the suitable conditions under which a four dimensional conformally flat Landsberg space becomes a Berwald space.

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1 Introduction

Let us consider a four-dimensional Finsler space $F^4 = (M, L)$ equipped with the fundamental function $L(x, y)$. The fundamental metric tensor g_{ij} and Cartan $C-$tensor C_{ijk} of F^4 are defined by ([2], [14])

(1.1) $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \qquad C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2.$

Throughout the paper, the symbols $\dot{\partial}_i = \frac{\partial}{\partial y^i}$ and $\partial_i = \frac{\partial}{\partial x^i}$ have been used. Let $(x, y) = \left(x^i, y^i\right)$ be a local coordinate system of the total space of the tangent bundle TM of the four- dimensional differentiable manifold M. If in a Finsler space there exists a local coordinate system $\left(x^i, y^i\right)$ in which the fundamental tensor g_{ij} can be written as a function of the variables y^i alone, the space is called locally Minkowski space and such a coordinate system is called a rectilinear coordinate system. If a Finsler space (M, L) is conformal to a locally Minkowski space $(M, \overline{L})$, then (M, L) is called a conformally flat Finsler space, more precisely, $\sigma-$ conformally flat Finsler space.

2 Scalar components in Miron frame

Let $F^4 = (M, L)$ be a four-dimensional Finsler space with fundamental function $L(x, y)$. The metric tensor g_{ij} and Cartan $C-$tensor C_{ijk} of F^4 are defined by (1.1). The frame $\left\{e^i_{(\alpha)}\right\}$ $(\alpha = 1, ..., 4)$ is called the Miron frame of F^4, where $e^i_{(1)} = l^i = \frac{y^i}{L}$ is called the normalized supporting element, $e^i_{(2)} = m^i = \frac{C^i}{C}$ is called the normalized torsion vector, $e^i_{(3)} = n^i, e^i_{(4)} = p^i$ are constructed by $g_{ij}e^i_{(\alpha)}e^j_{(\beta)} = \delta_{\alpha\beta}$. Here, C is the length of torsion vector $C_i = C_{ijk}g^{jk}$. The Greek letters $\alpha, \beta, \gamma, \delta$ vary from 1 to 4 throughout the paper. Summation convention is applied for both the Greek and Latin indices. In the Miron frame an arbitrary tensor can be expressed by scalar components along the unit vectors l^i, m^i, n^i, p^i. For instance; let $T = T^i_j$ be a tensor field of $(1, 1)$ type, then the scalar components $T_{\alpha\beta}$ of T are defined by $T_{\alpha\beta} = T^i_j e_{(\alpha)i}e^j_{(\beta)}$and the components T^i_j of the tensor T are expressed as $T^i_j = T_{\alpha\beta}e^i_{(\alpha)}e_{(\beta)j}$.
It is well known (See [18], [20]) that in a four-dimensional Finsler space there are eight main scalars $H,\ I,\ J,\ K,\ H',\ I',\ J',\ K'$ in which the sum of H, I and K is LC, which is called unified main scalar. In [20], it has been proved also that in a four-dimensional Finsler space there exist three h-connection vectors $h_i, j_i, k_i,$ whose scalar components with respect to the frame $\left\{e^i_{(\alpha)}\right\}$ are $h_\gamma, j_\gamma, k_\gamma$ i.e., $h_i = h_\gamma e_{(\gamma)i}, \quad j_i = j_\gamma e_{(\gamma)i}, k_i = k_\gamma e_{(\gamma)i}$. Further, in a four-dimensional Finsler space there exist three v-connection vectors u_i, v_i, w_i whose scalar components with respect to the frame $\left\{e^i_{(\alpha)}\right\}$ are $u_\gamma, v_\gamma, w_\gamma$ *i.e.*, $u_i = u_\gamma e_{(\gamma)i}, \qquad v_i = v_\gamma e_{(\gamma)i}, \qquad w_i = w_\gamma e_{(\gamma)i}$.

In the next sections, we use the following important results, obtained in [20]:

$$(2.1)\,(a)\ \ l_{i;j} = 0,\ m_{i;j} = n_i h_j + p_i j_j,\ n_{i;j} = -m_i h_j + p_i k_j,\ p_{i;j} = -m_i j_j - n_i k_j,$$

and

$$(2.1)\,(b)\ \ Ll_{i|j} = m_i m_j + n_i n_j + p_i p_j = g_{ij} - l_i l_j = h_{ij},$$

$$Lm_{i|j} = -l_i m_j + n_i u_j + p_i v_j, \qquad Ln_{i|j} = -l_i n_j - m_i u_j + p_i w_j,$$

$$Lp_{i|j} = -l_i p_j - m_i v_j - n_i w_j.$$

3 Conformal change of Cartan's connection

Let (M, L) and $\left(M, \bar{L}\right)$ be two Finsler manifolds. Then the two manifolds are conformal if there exists a positive differentiable function $\sigma : M \longrightarrow \mathbb{R}$ such that $\bar{L}(x, y) = e^{\sigma(x)}L(x, y)$. In this case, the transformation $L \longrightarrow \bar{L}$ is said to be conformal transformation and the Finsler manifolds (M, L) and $\left(M, \bar{L}\right)$ are said to be conformal or conformally related. If $\sigma(x) =$ constant, the transformation is called homothetic, otherwise, it is called non-homothetic transformation.
We consider a conformal change $L(x, y) \longrightarrow \overline{L}(x, y) = e^{\sigma(x)}L(x, y)$ of a four-dimensional Finsler space $F^4 = (M^4, L(x, y))$ with the fundamental function $L(x, y)$, where $\sigma(x)$ is a scalar function of position x^i alone, called the conformal factor. We shall denote the Finsler space with changed fundamental function $\overline{L}(x, y)$ by $\overline{F}^4 = (M^4,\ \overline{L}(x, y))$ and quantities of $\overline{F}^4$ by upper line. The following change of important quantities are well known ([6]).

(3.1) $\overline{l}_i = e^{\sigma} l_i,\ \overline{m}_i = e^{\sigma} m_i,\ \overline{n}_i = e^{\sigma} n_i,\ \overline{p}_i = e^{\sigma} p_i,\ \overline{g}_{ij} = e^{2\sigma} g_{ij},$

(3.2) $\overline{l}^i = e^{-\sigma} l^i,\ \overline{m}^i = e^{-\sigma} m^i,\ \overline{n}^i = e^{-\sigma} n^i,\ \overline{p}^i = e^{-\sigma} p^i,\ \overline{g}^{ij} = e^{-2\sigma} g^{ij},$

(3.3) $\overline{C}_{ijk} = e^{2\sigma} C_{ijk},\ \overline{C}^i_{jk} = C^i_{jk},\ \overline{H} = H,\ \overline{I} = I,\ \overline{J} = J,\ \overline{K} = K,\ \overline{H'} = H',\ \overline{I'} = I',$
$\overline{J'} = J',\ \overline{K'} = K'.$

In [20], it has been obtained that:

(3.4) $$\overline{G}^i_j = G^i_j + Ll^i\left(\sigma_1 l_j + \sigma_2 m_j + \sigma_3 n_j + \sigma_4 p_j\right) - Lm^i\left(\sigma_2 l_j - \sigma_5 m_j + \sigma_6 n_j + \sigma_7 p_j\right)$$
$$-Ln^i\left(\sigma_3 l_j + \sigma_6 m_j - \sigma_8 n_j - \sigma_9 p_j\right) - Lp^i\left(\sigma_4 l_j + \sigma_7 m_j - \sigma_9 n_j - \sigma_{10} p_j\right).$$

For the conformal change of the adapted components $h_\alpha, j_\alpha, k_\alpha$ of three h-connection vectors h_i, j_i, k_i, from (3.1) and (2.1)(a), we have [20]

(3.5) (a) $$\overline{h}_j = h_j + \left\{\sigma_2 u_2 + \sigma_3 u_3 + \sigma_4 u_4\right\} l_j$$
$$+\left\{-\sigma_5 u_2 + \sigma_6 u_3 + \sigma_7 u_4 + \sigma_5\left(J + J'\right) + \sigma_6 I + \sigma_7 K' - \sigma_{12}\right\} m_j$$
$$+\left\{\sigma_6 u_2 - \sigma_8 u_3 - \sigma_9 u_4 - \sigma_6\left(J + J'\right) - \sigma_8 I - \sigma_9 K' + \sigma_{14}\right\} n_j$$
$$+\left\{\sigma_7 u_2 - \sigma_9 u_3 - \sigma_{10} u_4 - \sigma_7\left(J + J'\right) - \sigma_9 I - \sigma_{10} K' + \sigma_{15}\right\} p_j,$$

(b) $$\overline{j}_j = j_j + \left\{\sigma_2 v_2 + \sigma_3 v_3 + \sigma_4 v_4\right\} l_j$$
$$+\left\{-\sigma_5 v_2 + \sigma_6 v_3 + \sigma_7 v_4 + \sigma_5\left(H' + I'\right) + \sigma_6 K' + \sigma_7 K - \sigma_{13}\right\} m_j$$
$$+\left\{\sigma_6 v_2 - \sigma_8 v_3 - \sigma_9 v_4 - \sigma_6\left(H' + I'\right) - \sigma_8 K' - \sigma_9 K + \sigma_{15}\right\} n_j$$
$$+\left\{\sigma_7 v_2 - \sigma_9 v_3 - \sigma_{10} v_4 - \sigma_7\left(H' + I'\right) - \sigma_9 K' - \sigma_{10} K + \sigma_{16}\right\} p_j,$$

(c) $$\overline{k}_j = k_j + \left\{\sigma_2 w_2 + \sigma_3 w_3 + \sigma_4 w_4\right\} l_j$$
$$+\left\{-\sigma_5 w_2 + \sigma_6 w_3 + \sigma_7 w_4 - \sigma_5 K' + \sigma_6 I' + \sigma_7 J' + \sigma_{19}\right\} m_j$$
$$+\left\{\sigma_6 w_2 - \sigma_8 w_3 - \sigma_9 w_4 + \sigma_6 K' - \sigma_8 I' - \sigma_9 J' - \sigma_{21}\right\} n_j$$
$$+\left\{\sigma_7 w_2 - \sigma_9 w_3 - \sigma_{10} w_4 + \sigma_7 K' - \sigma_9 I' - \sigma_{10} J' + \sigma_{19}\right\} p_j.$$

Thus the adapted components $\overline{h}_\alpha, \overline{j}_\alpha, \overline{k}_\alpha$ of $\overline{h}_i, \overline{j}_i, \overline{k}_i$ in $\overline{F}^4 = \left(M^4, \overline{L}(x,y)\right)$ are given by

(3.6) (a) $$\overline{h}_1 = e^{-\sigma}\left\{h_1 + \sigma_2 u_2 + \sigma_3 u_3 + \sigma_4 u_4\right\},$$

$$\overline{h}_2 = e^{-\sigma}\left\{h_2 - \sigma_5 u_2 + \sigma_6 u_3 + \sigma_7 u_4 + \sigma_5\left(J + J'\right) + \sigma_6 I + \sigma_7 K' - \sigma_{12}\right\},$$

$$\overline{h}_3 = e^{-\sigma}\left\{h_3 + \sigma_6 v_2 - \sigma_8 v_3 - \sigma_9 v_4 - \sigma_6\left(J + J'\right) - \sigma_8 I - \sigma_9 K' + \sigma_{14}\right\},$$

$$\overline{h}_4 = e^{-\sigma}\left\{h_4 + \sigma_7 u_2 - \sigma_9 u_3 - \sigma_{10} u_4 - \sigma_7\left(J + J'\right) - \sigma_9 I - \sigma_{10} K' + \sigma_{15}\right\}.$$

$$(b)\ \overline{j}_1 = e^{-\sigma}\left\{j_1 + \sigma_2 v_2 + \sigma_3 v_3 + \sigma_4 v_4\right\},$$

$$\overline{j}_2 = e^{-\sigma}\left\{j_2 - \sigma_5 v_2 + \sigma_6 v_3 + \sigma_7 v_4 + \sigma_5\left(H' + I'\right) + \sigma_6 K' + \sigma_7 K - \sigma_{13}\right\},$$

$$\overline{j}_3 = e^{-\sigma}\left\{j_3 + \sigma_6 v_2 - \sigma_8 v_3 - \sigma_9 v_4 - \sigma_6\left(H' + I'\right) - \sigma_8 K' - \sigma_9 K + \sigma_{15}\right\},$$

$$\overline{j}_4 = e^{-\sigma}\left\{j_4 + \sigma_7 v_2 - \sigma_9 v_3 - \sigma_{10} v_4 - \sigma_7\left(H' + I'\right) - \sigma_9 K' - \sigma_{10} K + \sigma_{16}\right\}.$$

$$(c)\ \overline{k}_1 = e^{-\sigma}\left\{k_1 + \sigma_2 w_2 + \sigma_3 w_3 + \sigma_4 w_4\right\},$$

$$\overline{k}_2 = e^{-\sigma}\left\{k_2 - \sigma_5 w_2 + \sigma_6 w_3 + \sigma_7 w_4 - \sigma_5 K' + \sigma_6 I' + \sigma_7 J' - \sigma_{19}\right\},$$

$$\overline{k}_3 = e^{-\sigma}\left\{k_3 + \sigma_6 w_2 - \sigma_8 w_3 - \sigma_9 w_4 + \sigma_6 K' - \sigma_8 I' - \sigma_9 J' - \sigma_{21}\right\},$$

$$\overline{k}_4 = e^{-\sigma}\left\{k_4 + \sigma_7 w_2 - \sigma_9 w_3 - \sigma_{10} w_4 + \sigma_7 K' - \sigma_9 I' - \sigma_{10} J' + \sigma_{19}\right\}.$$

4 Conformally flat Landsberg space

It is well known that the Berwald spaces are characterized by $C_{ijk|h} = 0$ and Landsberg spaces are characterized by $C_{ijk|0} = 0$ ([2], [14]), where the index '0' denotes the transvection by the supporting element y^i. Also, one knows that every Berwald space is a Landsberg space but the converse is not necessarily true. In ([13], [15], [21]), it has been shown that a Landsberg space becomes a Berwald space under some special conditions. A lot of work has been done on the conformal transformation and conformally flat manifolds in Euclidean, hyperbolic, Riemannian and Finsler geometry. Here, we mention some of the important results. In ([10], [11]), N. H. Kuiper studied conformally flat spaces in large and also for Euclidean spaces of dimension greater than 2. M. Gromov et. al [5], R. S. Kulkarni[12] have also worked on conformally flat manifolds. The notion of conformal transformation in Finsler spaces was introduced by M. Hashiguchi [6]. In particular, in dimension 4, some of the main contributions are due to M. F. Atiyah et. al [3], S. Y. Chang et.al [4], M. Iori et. al [7], M. Kalafat [8], and S. Akubulut et. al [1]. In [9], Michael Kapovich has discussed conformally flat metrics on Riemannian 4-manifolds, which itself says that there is a lot of possibility to extend these ideas in Finsler geometry. Motivated by these results, here we obtain the conditions under which a four-dimensional Landsberg space becomes a Berwald space. We use the following definition and theorems to obtain the required conditions.

Definition 1 [2]*A Finsler space F^n is called conformally flat if F^n is conformal to a locally Minkowski space.*

Theorem 2 [20] *A Finsler space F^4 with non-zero C is a Berwald space if and only if the h-connection vectors h_i, j_i, k_i vanish and all the main scalars are h-covariant constant.*

Theorem 3 [20]*A Finsler space F^4 with non-zero C is a Landsberg space if and only if the h-connection vectors h_i, j_i, k_i are orthogonal to the supporting element y^i i.e., $h_1 = j_1 = k_1 = 0$ and $H_{,1} = I_{,1} = J_{,1} = K_{,1} = H'_{,1} = I'_{,1} = J'_{,1} = K'_{,1} = 0$.*

If the four-dimensional Finsler space $\overline{F}^4 = \left(M, \overline{L}\right)$ is conformal to the Finsler space $F^4 = (M, L)$, the main scalars $\overline{H}, \overline{I}, \overline{J}, \overline{K}, \overline{H}', \overline{I}', \overline{J}', \overline{K}'$ of $\overline{F}^4$ coincide with the main scalars $H, I, J, K, H', I', J', K'$ of F^4. In particular we must notice that the main scalars $H, I, J, K, H', I', J', K'$ of F^4 and h-connection vectors h_i, j_i, k_i are functions of the variable y^i alone. Firstly, we assume that the Finsler space $\overline{F}^4 = \left(M, \overline{L}\right)$ is a Landsberg space. Then from Theorem 3, it follows that

$$(4.1)\ \ \overline{H}_{,1} = \overline{I}_{,1} = \overline{J}_{,1} = \overline{K}_{,1} = \overline{H}'_{,1} = \overline{I}'_{,1} = \overline{J}'_{,1} = \overline{K}'_{,1} = 0; \ \overline{h}_1 = \overline{j}_1 = \overline{k}_1 = 0.$$

The scalar $\overline{H}_{,1}$ can be written in terms of Miron frame as
$\overline{H}_{,1} = \overline{H}_{,k}\overline{l}^k = (\frac{\partial \overline{H}}{\partial x^k} - \overline{G}^r_k \frac{\partial \overline{H}}{\partial y^r})\overline{l}^k = -\overline{G}^r_k \frac{\partial \overline{H}}{\partial y^r}\overline{l}^k = -\overline{G}^r_k \frac{\partial H}{\partial y^r} e^{-\sigma} l^k$
Making use of (3.4), we get
$\overline{H}_{,1} = -Le^{-\sigma}(\sigma_1 H \mid_r l^r - \sigma_2 H \mid_r m^r - \sigma_3 H \mid_r n^r - \sigma_4 H \mid_r p^r).$
Therefore, we have

$$(4.2)\ \ \overline{H}_{,1} = (\sigma_2 H;_2 + \sigma_3 H;_3 + \sigma_4 H;_4)e^{-\sigma}, \overline{I}_{,1} = (\sigma_2 I;_2 + \sigma_3 I;_3 + \sigma_4 I;_4)e^{-\sigma},$$
$$\overline{J}_{,1} = (\sigma_2 J;_2 + \sigma_3 J;_3 + \sigma_4 J;_4)e^{-\sigma}, \overline{K}_{,1} = (\sigma_2 K;_2 + \sigma_3 K;_3 + \sigma_4 K;_4)e^{-\sigma},$$
$$\overline{H'}_{,1} = (\sigma_2 H';_2 + \sigma_3 H';_3 + \sigma_4 H';_4)e^{-\sigma}, \overline{I'}_{,1} = (\sigma_2 I';_2 + \sigma_3 I';_3 + \sigma_4 I';_4)e^{-\sigma},$$
$$\overline{J'}_{,1} = (\sigma_2 J';_2 + \sigma_3 J';_3 + \sigma_4 J';_4)e^{-\sigma}, \overline{K'}_{,1} = (\sigma_2 K';_2 + \sigma_3 K';_3 + \sigma_4 K';_4)e^{-\sigma}.$$

Therefore, from (4.1), (4.2) and (3.6), we get

$$(4.3)\ \ \sigma_2 H;_2 + \sigma_3 H;_3 + \sigma_4 H;_4 = 0, \qquad \sigma_2 I;_2 + \sigma_3 I;_3 + \sigma_4 I;_4 = 0,$$
$$\sigma_2 J;_2 + \sigma_3 J;_3 + \sigma_4 J;_4 = 0, \qquad \sigma_2 K;_2 + \sigma_3 K;_3 + \sigma_4 K;_4 = 0,$$
$$\sigma_2 H';_2 + \sigma_3 H';_3 + \sigma_4 H';_4 = 0, \qquad \sigma_2 I';_2 + \sigma_3 I';_3 + \sigma_4 I';_4 = 0,$$
$$\sigma_2 J';_2 + \sigma_3 J';_3 + \sigma_4 J';_4 = 0, \qquad \sigma_2 K';_2 + \sigma_3 K';_3 + \sigma_4 K';_4 = 0$$

and

$$h_1 + \sigma_2 u_2 + \sigma_3 u_3 + \sigma_4 u_4 = 0, \qquad j_1 + \sigma_2 v_2 + \sigma_3 v_3 + \sigma_4 v_4 = 0,$$
$$k_1 + \sigma_2 w_2 + \sigma_3 w_3 + \sigma_4 w_4 = 0.$$

Now, we prove that σ_2, σ_3 and σ_4 never vanish simultaneously, for non homothetic transformation. If possible let us assume that $\sigma_2 = 0, \sigma_3 = 0$ and $\sigma_4 = 0$, then $\sigma_i = \sigma_1 l_i + \sigma_2 m_i + \sigma_3 n_i + \sigma_4 p_i$ gives $\sigma_i = \sigma_1 l_i$. Differentiating this with respect to y^i, we get $0 = \left(\dot{\partial}_j \sigma_1\right) l_i + \sigma_1 \dot{\partial}_j l_i = \left(\dot{\partial}_j \sigma_1\right) l_i + \sigma_1 l_i \mid_j$, which in view of (2.1) (b) gives

(4.4) $\frac{\sigma_1}{L}\left(m_i m_j + n_i n_j + p_i p_j\right) = -\left(\dot{\partial}_j \sigma_1\right) l_i$

Since L. H. S. of the equation (4.4) is symmetric in i and j, we have $\left(\dot{\partial}_i \sigma_1\right) l_j = \left(\dot{\partial}_j \sigma_1\right) l_i$. Contracting this equation by l^j, we get $\dot{\partial}_i \sigma_1 = \left(\dot{\partial}_j \sigma_1\right) l_i l^j$. Since σ_1 is positively homogeneous of degree zero in y^i, we get $\left(\dot{\partial}_j \sigma_1\right) l^j = 0$, which implies $\dot{\partial}_i \sigma_1 = 0$. Thus, the equation (4.4) shows that $\sigma_1 = 0$. Hence $\sigma_i = 0$ $(i = 1, ..., 4)$; which shows that σ is constant, i.e., the transformation is homothetic. Hence, we conclude that, for a non-homothetic transformation σ_2, σ_3 and σ_4 never vanish simultaneously. So we consider the following cases for non-homothetic transformation.

Case(i) When $\sigma_2 \neq 0, \sigma_3 \neq 0,\ \sigma_4 \neq 0$.

In this case, we see that there is no change in equation (4.3) . From this, it follows that if the space is a Landsberg space then (4.3) holds.

Conversely, if (4.3) holds then from (4.2) and (3.6), we get (4.1). So, $\left(M, \overline{L}\right)$ is a Landsberg space.

Case(ii) When $\sigma_2 \neq 0, \sigma_3 \neq 0,\ \sigma_4 = 0$.

In this case, from the equation (4.3), we get

(4.5) $\frac{H_{;2}}{H_{;3}} = \frac{I_{;2}}{I_{;3}} = \frac{J_{;2}}{J_{;3}} = \frac{K_{;2}}{K_{;3}} = \frac{H'_{;2}}{H'_{;3}} = \frac{I'_{;2}}{I'_{;3}} = \frac{J'_{;2}}{J'_{;3}} = \frac{K'_{;2}}{K'_{;3}} = -\frac{\sigma_3}{\sigma_2}$

and

$$h_1 = -\left(\sigma_2 u_2 + \sigma_3 u_3\right), j_1 = -\left(\sigma_2 v_2 + \sigma_3 v_3\right), k_1 = -(\sigma_2 w_2 + \sigma_3 w_3).$$

Conversely, if (4.5) holds then from (3.6), (4.2) and (4.3), we get (4.1). So, (M, $\overline{L}$) is a Landsberg space.

Case(iii) When $\sigma_2 \neq 0, \sigma_3 = 0,\ \sigma_4 \neq 0$.

In this case, from the equation (4.3), we get

(4.6) $\frac{H_{;2}}{H_{;4}} = \frac{I_{;2}}{I_{;4}} = \frac{J_{;2}}{J_{;4}} = \frac{K_{;2}}{K_{;4}} = \frac{H'_{;2}}{H'_{;4}} = \frac{I'_{;2}}{I'_{;4}} = \frac{J'_{;2}}{J'_{;4}} = \frac{K'_{;2}}{K'_{;4}} = -\frac{\sigma_4}{\sigma_2}$

and

$$h_1 = -\left(\sigma_2 u_2 + \sigma_4 u_4\right), j_1 = -\left(\sigma_2 v_2 + \sigma_4 v_4\right), k_1 = -(\sigma_2 w_2 + \sigma_4 w_4).$$

Conversely, if (4.6) holds then from (3.6), (4.2) and (4.3), we get (4.1). So, (M, $\overline{L}$) is a Landsberg space.

Case(iv) When $\sigma_2 = 0, \sigma_3 \neq 0,\ \sigma_4 \neq 0$.

In this case, from the equation (4.3), we get

(4.7) $\frac{H_{;3}}{H_{;4}} = \frac{I_{;3}}{I_{;4}} = \frac{J_{;3}}{J_{;4}} = \frac{K_{;3}}{K_{;4}} = \frac{H'_{;3}}{H'_{;4}} = \frac{I'_{;3}}{I'_{;4}} = \frac{J'_{;3}}{J'_{;4}} = \frac{K'_{;3}}{K'_{;4}} = -\frac{\sigma_4}{\sigma_3}$

and

$$h_1 = -\left(\sigma_3 u_3 + \sigma_4 u_4\right), j_1 = -\left(\sigma_3 v_3 + \sigma_4 v_4\right), k_1 = -(\sigma_3 w_3 + \sigma_4 w_4).$$

Conversely, if (4.7) holds then from (3.6), (4.2) and (4.3), we get (4.1). So, $(M, \overline{L})$ is a Landsberg space.

Case(v) When $\sigma_2 \neq 0, \sigma_3$=0, $\sigma_4 = 0$.

In this case, from the equation (4.3), we get

(4.8) $H_{;2} = I_{;2} = J_{;2} = K_{;2} = H'_{;2} = I'_{;2} = J'_{;2} = K'_{;2} = 0$

and

$$h_1 = -\sigma_2 u_2, \qquad j_1 = -\sigma_2 v_2, \qquad k_1 = -\sigma_2 w_2.$$

Conversely, if (4.8) holds then from (3.6), (4.2) and (4.3), we get (4.1). So, (M, $\overline{L}$) is a Landsberg space.

Case(vi) When $\sigma_2 = 0, \sigma_3 \neq 0$, $\sigma_4 = 0$.

In this case, from the equation (4.3), we get

(4.9) $H_{;3} = I_{;3} = J_{;3} = K_{;3} = H'_{;3} = I'_{;3} = J'_{;3} = K'_{;3} = 0$

and

$$h_1 = -\sigma_3 u_3, \qquad j_1 = -\sigma_3 v_3, \qquad k_1 = -\sigma_3 w_3.$$

Conversely, if (4.9) holds then from (3.6), (4.2) and (4.3), we get (4.1). So, (M, $\overline{L}$) is a Landsberg space.

Case(vii) When $\sigma_2 = 0, \sigma_3 = 0$, $\sigma_4 \neq 0$.

In this case, from the equation (4.3), we get

(4.10) $H_{;4} = I_{;4} = J_{;4} = K_{;4} = H'_{;4} = I'_{;4} = J'_{;4} = K'_{;4} = 0$

and

$$h_1 = -\sigma_4 u_4, \qquad j_1 = -\sigma_4 v_4, \qquad k_1 = -\sigma_4 w_4.$$

Conversely, if (4.10) holds then from (3.6), (4.2) and (4.3), we get (4.1). So, (M, $\overline{L}$) is a Landsberg space. Hence we have the following:

Theorem 4 *A four-dimensional Landsberg space is $\sigma-$ conformally flat if and only if one of the following conditions is satisfied:*

1) the equation (4.3) holds for $\sigma_2 \neq 0, \sigma_3 \neq 0$, $\sigma_4 \neq 0$.

2) σ_i is orthogonal to p^i and the equation (4.5) holds for $\sigma_2 \neq 0, \sigma_3 \neq 0,\ \sigma_4 = 0$.

3) σ_i is orthogonal to n^i and the equation (4.6) holds for $\sigma_2 \neq 0, \sigma_3 = 0,\ \sigma_4 \neq 0$.

4) σ_i is orthogonal to m^i and the equation (4.7) holds for $\sigma_2 = 0, \sigma_3 \neq 0,\ \sigma_4 \neq 0$.

5) σ_i is orthogonal to n^i and p^i and the equation (4.8) holds for $\sigma_2 \neq 0, \sigma_3 = 0,\ \sigma_4 = 0$.

6) σ_i is orthogonal to m^i and p^i and the equation (4.9) holds for $\sigma_2 = 0, \sigma_3 \neq 0$, $\sigma_4 = 0$.

7) σ_i is orthogonal to m^i and n^i, and the equation (4.10) holds for $\sigma_2 = 0, \sigma_3 = 0$, $\sigma_4 \neq 0$.

5 Conformally flat Berwald space

A Finsler metric is called conformally Berwald Finsler metric if it is conformally related to a Berwald metric. In particular, if a Finsler metric is conformally related to a Minkowski metric, then it is called conformally flat Finsler metric. More precisely, a Finsler manifold (M, L) is called conformally flat or σ- conformally flat, if there exists an atlas whose changes of coordinates are conformal diffeomorphism to open sets in some Minkowski space. We consider the case when the Finsler space $\overline{F}^4 = \left(M, \overline{L}\right)$ is a Berwald space. We shall write $\overline{H}_{\perp k} = (\frac{\partial \overline{H}}{\partial x^k} - \overline{G}^r_k \frac{\partial \overline{H}}{\partial y^r})$. Since the main scalars $H, I, J, K, H', I', J', K'$ of F^4 and h-connection vectors h_i, j_i, k_i are functions of the variable y^i alone, the above equation is equivalent to $\overline{H}_{\perp k} = -\overline{G}^r_k \frac{\partial H}{\partial y^r}$. Making use of (3.4), we get

$$(5.1)\quad \overline{H}_{\perp k} = -L\{l^r\left(\sigma_1 l_k + \sigma_2 m_k + \sigma_3 n_k + \sigma_4 p_k\right) - m^r\left(\sigma_2 l_k - \sigma_5 m_k + \sigma_6 n_k + \sigma_7 p_k\right)$$
$$-n^r\left(\sigma_3 l_k + \sigma_6 m_k - \sigma_8 n_k - \sigma_9 p_k\right) - p^r\left(\sigma_4 l_k + \sigma_7 m_k - \sigma_9 n_k - \sigma_{10} p_k\right)\} H\,|_r\ .$$

Since $H\,|_r = L^{-1}(H;_1 l_r + H;_2 m_r + H;_3 n_r + H;_4 p_r)$ and $H;_1 = 0$, we have

$$(5.2)\quad \overline{H}_{\perp k} = H;_2\left(\sigma_2 l_k - \sigma_5 m_k + \sigma_6 n_k + \sigma_7 p_k\right) + H;_3\left(\sigma_3 l_k + \sigma_6 m_k - \sigma_8 n_k - \sigma_9 p_k\right)$$
$$+H;_4\left(\sigma_6 l_k + \sigma_7 m_k - \sigma_9 n_k - \sigma_{10} p_k\right),$$
$$\overline{I}_{\perp k} = I;_2\left(\sigma_2 l_k - \sigma_5 m_k + \sigma_6 n_k + \sigma_7 p_k\right) + I;_3\left(\sigma_3 l_k + \sigma_6 m_k - \sigma_8 n_k - \sigma_9 p_k\right)$$
$$+I;_4\left(\sigma_6 l_k + \sigma_7 m_k - \sigma_9 n_k - \sigma_{10} p_k\right),$$
$$\overline{J}_{\perp k} = J;_2\left(\sigma_2 l_k - \sigma_5 m_k + \sigma_6 n_k + \sigma_7 p_k\right) + J;_3\left(\sigma_3 l_k + \sigma_6 m_k - \sigma_8 n_k - \sigma_9 p_k\right)$$
$$+J;_4\left(\sigma_6 l_k + \sigma_7 m_k - \sigma_9 n_k - \sigma_{10} p_k\right),$$
$$\overline{K}_{\perp k} = K;_2\left(\sigma_2 l_k - \sigma_5 m_k + \sigma_6 n_k + \sigma_7 p_k\right) + K;_3\left(\sigma_3 l_k + \sigma_6 m_k - \sigma_8 n_k - \sigma_9 p_k\right)$$
$$+K;_4\left(\sigma_6 l_k + \sigma_7 m_k - \sigma_9 n_k - \sigma_{10} p_k\right),$$
$$\overline{H'}_{\perp k} = H';_2\left(\sigma_2 l_k - \sigma_5 m_k + \sigma_6 n_k + \sigma_7 p_k\right) + H';_3\left(\sigma_3 l_k + \sigma_6 m_k - \sigma_8 n_k - \sigma_9 p_k\right)$$

$$+H';_4\left(\sigma_6 l_k+\sigma_7 m_k-\sigma_9 n_k-\sigma_{10} p_k\right),$$

$$\overline{I'}_{\perp k}=I';_2\left(\sigma_2 l_k-\sigma_5 m_k+\sigma_6 n_k+\sigma_7 p_k\right)+I';_3\left(\sigma_3 l_k+\sigma_6 m_k-\sigma_8 n_k-\sigma_9 p_k\right)$$

$$+I';_4\left(\sigma_6 l_k+\sigma_7 m_k-\sigma_9 n_k-\sigma_{10} p_k\right),$$

$$\overline{J'}_{\perp k}=J';_2\left(\sigma_2 l_k-\sigma_5 m_k+\sigma_6 n_k+\sigma_7 p_k\right)+J';_3\left(\sigma_3 l_k+\sigma_6 m_k-\sigma_8 n_k-\sigma_9 p_k\right)$$

$$+J';_4\left(\sigma_6 l_k+\sigma_7 m_k-\sigma_9 n_k-\sigma_{10} p_k\right),$$

$$\overline{K'}_{\perp k}=K';_2\left(\sigma_2 l_k-\sigma_5 m_k+\sigma_6 n_k+\sigma_7 p_k\right)+K';_3\left(\sigma_3 l_k+\sigma_6 m_k-\sigma_8 n_k-\sigma_9 p_k\right)$$

$$+K';_4\left(\sigma_6 l_k+\sigma_7 m_k-\sigma_9 n_k-\sigma_{10} p_k\right).$$

Now, we discuss all the seven cases which have been discussed in the previous section.

Case(i) When $\sigma_2 \neq 0, \sigma_3 \neq 0, \sigma_4 \neq 0$.

If the four dimensional Landsberg space $\left(M, \overline{L}\right)$ is conformally flat, then from the equations (5.2), (3.5) and (4.3), we get

$$(5.3)\quad \overline{H}_{\perp k}=\left(-\sigma_5 H;_2+\sigma_6 H;_3+\sigma_7 H;_4\right) m_k+\left(\sigma_6 H;_2-\sigma_8 H;_3-\sigma_9 H;_4\right) n_k$$

$$+\left(\sigma_7 H;_2-\sigma_9 H;_3-\sigma_{10} H;_4\right) p_k,$$

$$\overline{I}_{\perp k}=\left(-\sigma_5 I;_2+\sigma_6 I;_3+\sigma_7 I;_4\right) m_k+\left(\sigma_6 I;_2-\sigma_8 I;_3-\sigma_9 I;_4\right) n_k$$

$$+\left(\sigma_7 I;_2-\sigma_9 I;_3-\sigma_{10} I;_4\right) p_k,$$

$$\overline{J}_{\perp k}=\left(-\sigma_5 J;_2+\sigma_6 J;_3+\sigma_7 J;_4\right) m_k+\left(\sigma_6 J;_2-\sigma_8 J;_3-\sigma_9 J;_4\right) n_k$$

$$+\left(\sigma_7 J;_2-\sigma_9 J;_3-\sigma_{10} J;_4\right) p_k,$$

$$\overline{K}_{\perp k}=\left(-\sigma_5 K;_2+\sigma_6 K;_3+\sigma_7 K;_4\right) m_k+\left(\sigma_6 K;_2-\sigma_8 K;_3-\sigma_9 K;_4\right) n_k$$

$$+\left(\sigma_7 K;_2-\sigma_9 K;_3-\sigma_{10} K;_4\right) p_k,$$

$$\overline{H'}_{\perp k}=\left(-\sigma_5 H';_2+\sigma_6 H';_3+\sigma_7 H';_4\right) m_k+\left(\sigma_6 H';_2-\sigma_8 H';_3-\sigma_9 H';_4\right) n_k$$

$$+\left(\sigma_7 H';_2-\sigma_9 H';_3-\sigma_{10} H';_4\right) p_k,$$

$$\overline{I'}_{\perp k}=\left(-\sigma_5 I';_2+\sigma_6 I';_3+\sigma_7 I';_4\right) m_k+\left(\sigma_6 I';_2-\sigma_8 I';_3-\sigma_9 I';_4\right) n_k$$

$$+\left(\sigma_7 I';_2-\sigma_9 I';_3-\sigma_{10} I';_4\right) p_k,$$

$$\overline{J'}_{\perp k}=\left(-\sigma_5 J';_2+\sigma_6 J';_3+\sigma_7 J';_4\right) m_k+\left(\sigma_6 J';_2-\sigma_8 J';_3-\sigma_9 J';_4\right) n_k$$

$$+\left(\sigma_7 J';_2-\sigma_9 J';_3-\sigma_{10} K';_4\right) p_k,$$

$$\overline{K'}_{\perp k}=\left(-\sigma_5 K';_2+\sigma_6 K';_3+\sigma_7 K';_4\right) m_k+\left(\sigma_6 K';_2-\sigma_8 K';_3-\sigma_9 K';_4\right) n_k$$

$$+\left(\sigma_7 K';_2-\sigma_9 K';_3-\sigma_{10} K';_4\right) p_k.$$

and

$$\overline{h}_j = [h_2 + \{-\sigma_5 u_2 + \sigma_6 u_3 + \sigma_7 u_4 + \sigma_5 (J + J') + \sigma_6 I + \sigma_7 K' - \sigma_{12}\}]m_j$$
$$+[h_3 + \{\sigma_6 u_2 - \sigma_8 u_3 - \sigma_9 u_4 - \sigma_6 (J + J') - \sigma_8 I - \sigma_9 K' + \sigma_{14}\}]n_j$$
$$+[h_4 + \{\sigma_7 u_2 - \sigma_9 u_3 - \sigma_{10} u_4 - \sigma_7 (J + J') - \sigma_9 I - \sigma_{10} K' + \sigma_{15}\}]p_j,$$
$$\overline{j}_j = [j_2 + \{-\sigma_5 v_2 + \sigma_6 v_3 + \sigma_7 v_4 + \sigma_5 (H' + I') + \sigma_6 K' + \sigma_7 K - \sigma_{13}\}]m_j$$
$$+[j_3 + \{\sigma_6 v_2 - \sigma_8 v_3 - \sigma_9 v_4 - \sigma_6 (H' + I') - \sigma_8 K' - \sigma_9 K + \sigma_{15}\}]n_j$$
$$+[j_4 + \{\sigma_7 v_2 - \sigma_9 v_3 - \sigma_{10} v_4 - \sigma_7 (H' + I') - \sigma_9 K' - \sigma_{10} K + \sigma_{16}\}]p_j,$$
$$\overline{k}_j = [k_2 + \{-\sigma_5 w_2 + \sigma_6 w_3 + \sigma_7 w_4 - \sigma_5 K' + \sigma_6 I' + \sigma_7 J' + \sigma_{19}\}]m_j$$
$$+[k_3 + \{\sigma_6 w_2 - \sigma_8 w_3 - \sigma_9 w_4 + \sigma_6 K' - \sigma_8 I' - \sigma_9 J' - \sigma_{21}\}]n_j$$
$$+[k_4 + \{\sigma_7 w_2 - \sigma_9 w_3 - \sigma_{10} w_4 + \sigma_7 K' - \sigma_9 I' - \sigma_{10} J' + \sigma_{19}\}]p_j.$$

From Theorem 2, it follows that the space $\mathrm{F}^4 = \left(M, \overline{L}\right)$ is a Berwald space if $\overline{H}_{\perp k} = \overline{I}_{\perp k} = \overline{J}_{\perp k} = \overline{K}_{\perp k} = \overline{H}'_{\perp k} = \overline{I'}_{\perp k} = \overline{J'}_{\perp k} = \overline{K'}_{\perp k} = 0$ and $\overline{h_i} = \overline{j_i} = \overline{k_i} = 0$. Therefore from (5.3), it follows that the space $\left(M, \overline{L}\right)$ is a Berwald space if

$$(5.4)\quad -\sigma_5 H;_2 + \sigma_6 H;_3 + \sigma_7 H;_4 = 0, \qquad \sigma_6 H;_2 - \sigma_8 H;_3 - \sigma_9 H;_4 = 0,$$
$$\sigma_7 H;_2 - \sigma_9 H;_3 - \sigma_{10} H;_4 = 0, \qquad -\sigma_5 I;_2 + \sigma_6 I;_3 + \sigma_7 I;_4 = 0,$$
$$\sigma_6 I;_2 - \sigma_8 I;_3 - \sigma_9 I;_4 = 0, \qquad \sigma_7 I;_2 - \sigma_9 I;_3 - \sigma_{10} I;_4 = 0,$$
$$-\sigma_5 J;_2 + \sigma_6 J;_3 + \sigma_7 J;_4 = 0, \qquad \sigma_6 J;_2 - \sigma_8 J;_3 - \sigma_9 J;_4 = 0,$$
$$\sigma_7 J;_2 - \sigma_9 J;_3 - \sigma_{10} J;_4 = 0, \qquad -\sigma_5 K;_2 + \sigma_6 K;_3 + \sigma_7 K;_4 = 0,$$
$$\sigma_6 K;_2 - \sigma_8 K;_3 - \sigma_9 K;_4 = 0, \qquad \sigma_7 K;_2 - \sigma_9 K;_3 - \sigma_{10} K;_4 = 0,$$
$$-\sigma_5 H';_2 + \sigma_6 H';_3 + \sigma_7 H';_4 = 0, \qquad \sigma_6 H';_2 - \sigma_8 H';_3 - \sigma_9 H';_4 = 0,$$
$$\sigma_7 H';_2 - \sigma_9 H';_3 - \sigma_{10} H';_4 = 0, \qquad -\sigma_5 I';_2 + \sigma_6 I';_3 + \sigma_7 I';_4 = 0,$$
$$\sigma_6 I';_2 - \sigma_8 I';_3 - \sigma_9 I';_4 = 0, \qquad \sigma_7 I';_2 - \sigma_9 I';_3 - \sigma_{10} I';_4 = 0,$$
$$-\sigma_5 J';_2 + \sigma_6 J';_3 + \sigma_7 J';_4 = 0, \qquad \sigma_6 J';_2 - \sigma_8 J';_3 - \sigma_9 J';_4 = 0,$$
$$\sigma_7 J';_2 - \sigma_9 J';_3 - \sigma_{10} K';_4 = 0, \qquad -\sigma_5 K';_2 + \sigma_6 K';_3 + \sigma_7 K';_4 = 0,$$
$$\sigma_6 K';_2 - \sigma_8 K';_3 - \sigma_9 K';_4 = 0, \qquad \sigma_7 K';_2 - \sigma_9 K';_3 - \sigma_{10} K';_4 = 0.$$

and

$$h_2 + \{-\sigma_5 u_2 + \sigma_6 u_3 + \sigma_7 u_4 + \sigma_5 (J + J') + \sigma_6 I + \sigma_7 K' - \sigma_{12}\} = 0,$$
$$h_3 + \{\sigma_6 u_2 - \sigma_8 u_3 - \sigma_9 u_4 - \sigma_6 (J + J') - \sigma_8 I - \sigma_9 K' + \sigma_{14}\} = 0,$$

$$h_4+\{\sigma_7 u_2-\sigma_9 u_3-\sigma_{10} u_4-\sigma_7\left(J+J'\right)-\sigma_9 I-\sigma_{10} K'+\sigma_{15}\}=0,$$

$$j_2+\{-\sigma_5 v_2+\sigma_6 v_3+\sigma_7 v_4+\sigma_5\left(H'+I'\right)+\sigma_6 K'+\sigma_7 K-\sigma_{13}\}=0,$$

$$j_3+\{\sigma_6 v_2-\sigma_8 v_3-\sigma_9 v_4-\sigma_6\left(H'+I'\right)-\sigma_8 K'-\sigma_9 K+\sigma_{15}\}=0,$$

$$j_4+\{\sigma_7 v_2-\sigma_9 v_3-\sigma_{10} v_4-\sigma_7\left(H'+I'\right)-\sigma_9 K'-\sigma_{10} K+\sigma_{16}\}=0,$$

$$k_2+\{-\sigma_5 w_2+\sigma_6 w_3+\sigma_7 w_4-\sigma_5 K'+\sigma_6 I'+\sigma_7 J'+\sigma_{19}\}=0,$$

$$k_3+\{\sigma_6 w_2-\sigma_8 w_3-\sigma_9 w_4+\sigma_6 K'-\sigma_8 I'-\sigma_9 J'-\sigma_{21}\}=0,$$

$$k_4+\{\sigma_7 w_2-\sigma_9 w_3-\sigma_{10} w_4+\sigma_7 K'-\sigma_9 I'-\sigma_{10} J'+\sigma_{19}\}=0.$$

Conversely, if (5.4) holds, then from (5.3), we get $\overline{H}_{\perp k}=\overline{I}_{\perp k}=\overline{J}_{\perp k}=\overline{K}_{\perp k}=\overline{H}'_{\perp k}=\overline{I'}_{\perp k}=\overline{J'}_{\perp k}=\overline{K'}_{\perp k}=0$ and $\overline{h_i}=\overline{j_i}=\overline{k_i}=0$. Hence the space $\left(M,\overline{L}\right)$ is a Berwald space. In the cases

(ii) When $\sigma_2\neq 0,\sigma_3\neq 0,\ \sigma_4=0,$

(iii) When $\sigma_2\neq 0,\sigma_3=0,\ \sigma_4\neq 0,$ and

(iv) When $\sigma_2=0,\sigma_3\neq 0,\ \sigma_4\neq 0,$

we see that if a four- dimensional Landsberg space is $\sigma-$conformally flat, then from the equations (5.2), (3.5) and (4.5), there is no change in the equation (5.4).

Case(v) When $\sigma_2\neq 0,\sigma_3=0,\ \sigma_4=0.$

In this case, if a four- dimensional Landsberg space is $\sigma-$conformally flat, then from the equations (5.2), (3.5) and (4.8), we see that (5.4) reduces to

$$\text{(5.5)}\quad \tfrac{H_{;3}}{H_{;4}}=\tfrac{I_{;3}}{I_{;4}}=\tfrac{J_{;3}}{J_{;4}}=\tfrac{K_{;3}}{K_{;4}}=\tfrac{H'_{;3}}{H'_{;4}}=\tfrac{I'_{;3}}{I'_{;4}}=\tfrac{J'_{;3}}{J'_{;4}}=\tfrac{K'_{;3}}{K'_{;4}}=-\tfrac{\sigma_7}{\sigma_6}=-\tfrac{\sigma_9}{\sigma_8}=-\tfrac{\sigma_{10}}{\sigma_9}$$

and

$$h_2+\{-\sigma_5 u_2+\sigma_6 u_3+\sigma_7 u_4+\sigma_5\left(J+J'\right)+\sigma_6 I+\sigma_7 K'-\sigma_{12}\}=0,$$

$$h_3+\{\sigma_6 u_2-\sigma_8 u_3-\sigma_9 u_4-\sigma_6\left(J+J'\right)-\sigma_8 I-\sigma_9 K'+\sigma_{14}\}=0,$$

$$h_4+\{\sigma_7 u_2-\sigma_9 u_3-\sigma_{10} u_4-\sigma_7\left(J+J'\right)-\sigma_9 I-\sigma_{10} K'+\sigma_{15}\}=0,$$

$$j_2+\{-\sigma_5 v_2+\sigma_6 v_3+\sigma_7 v_4+\sigma_5\left(H'+I'\right)+\sigma_6 K'+\sigma_7 K-\sigma_{13}\}=0,$$

$$j_3+\{\sigma_6 v_2-\sigma_8 v_3-\sigma_9 v_4-\sigma_6\left(H'+I'\right)-\sigma_8 K'-\sigma_9 K+\sigma_{15}\}=0,$$

$$j_4+\{\sigma_7 v_2-\sigma_9 v_3-\sigma_{10} v_4-\sigma_7\left(H'+I'\right)-\sigma_9 K'-\sigma_{10} K+\sigma_{16}\}=0,$$

$$k_2+\{-\sigma_5 w_2+\sigma_6 w_3+\sigma_7 w_4-\sigma_5 K'+\sigma_6 I'+\sigma_7 J'+\sigma_{19}\}=0,$$

$$k_3+\{\sigma_6 w_2-\sigma_8 w_3-\sigma_9 w_4+\sigma_6 K'-\sigma_8 I'-\sigma_9 J'-\sigma_{21}\}=0,$$

$$k_4+\{\sigma_7 w_2-\sigma_9 w_3-\sigma_{10} w_4+\sigma_7 K'-\sigma_9 I'-\sigma_{10} J'+\sigma_{19}\}=0.$$

Conversely, if (5.5) holds, then from (5.3), we get $\overline{H}_{\perp k}=\overline{I}_{\perp k}=\overline{J}_{\perp k}=\overline{K}_{\perp k}=\overline{H}'_{\perp k}=\overline{I'}_{\perp k}=\overline{J'}_{\perp k}=\overline{K'}_{\perp k}=0$ and $\overline{h_i}=\overline{j_i}=\overline{k_i}=0$. Thus the space $\left(M,\overline{L}\right)$ is a Berwald space.

Case(vi) When $\sigma_2 = 0, \sigma_3 \neq 0$, $\sigma_4 = 0$.

In this case, if a four- dimensional Landsberg space is $\sigma-$conformally flat, then from the equations (5.2), (3.5) and (4.9), we see that (5.4) reduces to

$$(5.6)\quad \frac{H_{;2}}{H_{;4}} = \frac{I_{;2}}{I_{;4}} = \frac{J_{;2}}{J_{;4}} = \frac{K_{;2}}{K_{;4}} = \frac{H'_{;2}}{H'_{;4}} = \frac{I'_{;2}}{I'_{;4}} = \frac{J'_{;2}}{J'_{;4}} = \frac{K'_{;2}}{K'_{;4}} = \frac{\sigma_7}{\sigma_5} = \frac{\sigma_9}{\sigma_6} = \frac{\sigma_{10}}{\sigma_7}$$

and

$$h_2 + \{-\sigma_5 u_2 + \sigma_6 u_3 + \sigma_7 u_4 + \sigma_5 (J + J') + \sigma_6 I + \sigma_7 K' - \sigma_{12}\} = 0,$$

$$h_3 + \{\sigma_6 u_2 - \sigma_8 u_3 - \sigma_9 u_4 - \sigma_6 (J + J') - \sigma_8 I - \sigma_9 K' + \sigma_{14}\} = 0,$$

$$h_4 + \{\sigma_7 u_2 - \sigma_9 u_3 - \sigma_{10} u_4 - \sigma_7 (J + J') - \sigma_9 I - \sigma_{10} K' + \sigma_{15}\} = 0,$$

$$j_2 + \{-\sigma_5 v_2 + \sigma_6 v_3 + \sigma_7 v_4 + \sigma_5 (H' + I') + \sigma_6 K' + \sigma_7 K - \sigma_{13}\} = 0,$$

$$j_3 + \{\sigma_6 v_2 - \sigma_8 v_3 - \sigma_9 v_4 - \sigma_6 (H' + I') - \sigma_8 K' - \sigma_9 K + \sigma_{15}\} = 0,$$

$$j_4 + \{\sigma_7 v_2 - \sigma_9 v_3 - \sigma_{10} v_4 - \sigma_7 (H' + I') - \sigma_9 K' - \sigma_{10} K + \sigma_{16}\} = 0,$$

$$k_2 + \{-\sigma_5 w_2 + \sigma_6 w_3 + \sigma_7 w_4 - \sigma_5 K' + \sigma_6 I' + \sigma_7 J' + \sigma_{19}\} = 0,$$

$$k_3 + \{\sigma_6 w_2 - \sigma_8 w_3 - \sigma_9 w_4 + \sigma_6 K' - \sigma_8 I' - \sigma_9 J' - \sigma_{21}\} = 0,$$

$$k_4 + \{\sigma_7 w_2 - \sigma_9 w_3 - \sigma_{10} w_4 + \sigma_7 K' - \sigma_9 I' - \sigma_{10} J' + \sigma_{19}\} = 0.$$

Conversely, if (5.6) holds, then from (5.3), we get $\overline{H}_{\perp k} = \overline{I}_{\perp k} = \overline{J}_{\perp k} = \overline{K}_{\perp k} = \overline{H'}_{\perp k} = \overline{I'}_{\perp k} = \overline{J'}_{\perp k} = \overline{K'}_{\perp k} = 0$ and $\overline{h_i} = \overline{j_i} = \overline{k_i} = 0$. Thus the space $\left(M, \overline{L}\right)$ is a Berwald space.

Case(vii) When $\sigma_2 = 0, \sigma_3$=0, $\sigma_4 \neq 0$.

In this case, if a four- dimensional Landsberg space is $\sigma-$conformally flat, then from the equations (5.2), (3.5) and (4.10), we see that (5.4) reduces to

$$(5.7)\quad \frac{H_{;2}}{H_{;3}} = \frac{I_{;2}}{I_{;3}} = \frac{J_{;2}}{J_{;3}} = \frac{K_{;2}}{K_{;3}} = \frac{H'_{;2}}{H'_{;3}} = \frac{I'_{;2}}{I'_{;3}} = \frac{J'_{;2}}{J'_{;3}} = \frac{K'_{;2}}{K'_{;3}} = \frac{\sigma_6}{\sigma_5} = \frac{\sigma_8}{\sigma_6} = \frac{\sigma_9}{\sigma_7}$$

and

$$h_2 + \{-\sigma_5 u_2 + \sigma_6 u_3 + \sigma_7 u_4 + \sigma_5 (J + J') + \sigma_6 I + \sigma_7 K' - \sigma_{12}\} = 0,$$

$$h_3 + \{\sigma_6 u_2 - \sigma_8 u_3 - \sigma_9 u_4 - \sigma_6 (J + J') - \sigma_8 I - \sigma_9 K' + \sigma_{14}\} = 0,$$

$$h_4 + \{\sigma_7 u_2 - \sigma_9 u_3 - \sigma_{10} u_4 - \sigma_7 (J + J') - \sigma_9 I - \sigma_{10} K' + \sigma_{15}\} = 0,$$

$$j_2 + \{-\sigma_5 v_2 + \sigma_6 v_3 + \sigma_7 v_4 + \sigma_5 (H' + I') + \sigma_6 K' + \sigma_7 K - \sigma_{13}\} = 0,$$

$$j_3 + \{\sigma_6 v_2 - \sigma_8 v_3 - \sigma_9 v_4 - \sigma_6 (H' + I') - \sigma_8 K' - \sigma_9 K + \sigma_{15}\} = 0,$$

$$j_4 + \{\sigma_7 v_2 - \sigma_9 v_3 - \sigma_{10} v_4 - \sigma_7 (H' + I') - \sigma_9 K' - \sigma_{10} K + \sigma_{16}\} = 0,$$

$$k_2 + \{-\sigma_5 w_2 + \sigma_6 w_3 + \sigma_7 w_4 - \sigma_5 K' + \sigma_6 I' + \sigma_7 J' + \sigma_{19}\} = 0,$$

$$k_3 + \{\sigma_6 w_2 - \sigma_8 w_3 - \sigma_9 w_4 + \sigma_6 K' - \sigma_8 I' - \sigma_9 J' - \sigma_{21}\} = 0,$$

$$k_4 + \{\sigma_7 w_2 - \sigma_9 w_3 - \sigma_{10} w_4 + \sigma_7 K' - \sigma_9 I' - \sigma_{10} J' + \sigma_{19}\} = 0.$$

Conversely, if (5.7) holds, then from (5.3), we get $\overline{H}_{\perp k} = \overline{I}_{\perp k} = \overline{J}_{\perp k} = \overline{K}_{\perp k} = \overline{H}'_{\perp k} = \overline{I'}_{\perp k} = \overline{J'}_{\perp k} = \overline{K'}_{\perp k} = 0$ and $\overline{h_i} = \overline{j_i} = \overline{k_i} = 0$. Thus the space $\big(M, \overline{L}\big)$ is a Berwald space. Hence we have the following:

Theorem 5 *A four-dimensional conformally flat Landsberg space is a Berwald space if and only if one of the following conditions is satisfied:*

1) *the equations (5.4) hold for* $\sigma_2 \neq 0, \sigma_3 \neq 0$, $\sigma_4 \neq 0$.

2) σ_i *is orthogonal to* p^i *and the equations (5.4) hold for* $\sigma_2 \neq 0, \sigma_3 \neq 0$, $\sigma_4 = 0$.

3) σ_i *is orthogonal to* n^i *and the equations (5.4) hold for* $\sigma_2 \neq 0, \sigma_3 = 0$, $\sigma_4 \neq 0$.

4) σ_i *is orthogonal to* m^i *and the equations (5.4) hold for* $\sigma_2 = 0, \sigma_3 \neq 0$, $\sigma_4 \neq 0$.

5) σ_i *is orthogonal to* n^i *and* p^i, *and the equations (5.5) hold for* $\sigma_2 \neq 0, \sigma_3 = 0$, $\sigma_4 = 0$.

6) σ_i *is orthogonal to* m^i *and* p^i, *and the equations (5.6) hold for* $\sigma_2 = 0, \sigma_3 \neq 0$, $\sigma_4 = 0$.

7) σ_i *is orthogonal to* m^i *and* n^i, *the equations (5.7) hold for* $\sigma_2 = 0, \sigma_3 = 0$, $\sigma_4 \neq 0$.

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