

Comment on “Lie symmetries and group classification of a class of time fractional evolution systems”

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In this article, the prolongation formulae proposed in Huang and Shen [J. Math. Phys. **56**, 123504 (2015)] for the Lie symmetry analysis of time fractional systems of partial differential equations are proved incomplete and the required correct prolongation operators are suggested. With the help of some examples, the efficiency of the operators introduced in this study is illustrated. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4983050>]

I. INTRODUCTION

Lie symmetry method is an efficient tool for studying fractional partial differential equations^{2,7,9} (FPDEs) with Riemann-Liouville fractional derivative.⁴ Huang and Shen³ extended the Lie symmetry approach for solving time fractional systems of partial differential equations (PDEs). The fractional order prolongation operators have been proposed to allow the determination of Lie symmetries for systems of FPDEs and applied to a time fractional system leading to its similarity reductions and some exact solutions. In this study, it is shown that their prolongation formulae³ are incomplete and are not able to derive the general form of Lie symmetries for time fractional systems. Therefore, the required modifications in the formulae are suggested in this work to overcome the shortcomings of their formulae. The main aim of the study is to introduce the correct general prolongation formulae for providing generalized Lie symmetries for time fractional systems of PDEs with an arbitrary number of dependent variables. For justification of our claim, both kinds of formulae are applied for the derivation of Lie point symmetries for some time fractional systems.

The fractional order prolongation formulae $\eta^{i(\alpha,0)}$ ($i = 1, \dots, m$) provided in the Ref. 3 for the time fractional systems of PDEs with m dependent variables (u^1, \dots, u^m) are as follows:

$$\begin{aligned} \eta^{i(\alpha,0)} = & \frac{\partial^\alpha \eta^i}{\partial t^\alpha} + \left(\frac{\partial \eta^i}{\partial u^i} - \alpha D_t \tau \right) \frac{\partial^\alpha u^i}{\partial t^\alpha} - u^i \frac{\partial^\alpha}{\partial t^\alpha} \left(\frac{\partial \eta^i}{\partial u^i} \right) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x^i) \\ & + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^{n+1} \eta^i}{\partial t^n \partial u^i} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n} u^i + \mu^i. \end{aligned} \quad (1)$$

In our point of view, formulae (1) are incomplete and the correct general prolongation operators are proposed in the following theorem:

Theorem 1. *The general prolongation formulae $\eta^{i(\alpha,0)}$ ($i = 1, \dots, m$) for time fractional systems of PDEs having m dependent variables (u^1, \dots, u^m) and independent variables (x, t) for Riemann-Liouville fractional derivative $0 < \alpha < 1$ are as follows:*

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$$\begin{aligned} \eta^{i(\alpha,0)} &= \frac{\partial^\alpha \eta^i}{\partial t^\alpha} + \left(\frac{\partial \eta^i}{\partial u^i} - \alpha D_t \tau \right) \frac{\partial^\alpha u^i}{\partial t^\alpha} - u^i \frac{\partial^\alpha}{\partial t^\alpha} \left(\frac{\partial \eta^i}{\partial u^i} \right) + \sum_{j \neq i} \left(\frac{\partial \eta^i}{\partial u^j} \frac{\partial^\alpha u^j}{\partial t^\alpha} - u^j \frac{\partial^\alpha}{\partial t^\alpha} \left(\frac{\partial \eta^i}{\partial u^j} \right) \right) \\ &\quad - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x^i) + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^{n+1} \eta^i}{\partial t^n \partial u^i} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n} u^i \\ &\quad + \sum_{j \neq i} \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^{n+1} \eta^i}{\partial t^n \partial u^j} D_t^{\alpha-n}(u^j) + \sum_{p=1}^m \mu_p^i, \end{aligned} \tag{2}$$

where

$$\mu_p^i = \sum_{n=2}^{\infty} \sum_{q=2}^n \sum_{k=2}^q \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{q} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)} (-u^p)^r \frac{\partial^q}{\partial t^q} (u^p)^{k-r} \frac{\partial^{n-q+k} \eta^i}{\partial t^{n-q} (\partial u^p)^k}. \tag{3}$$

It is important to note that the expressions for μ_p^i become zero if the η^i are linear in the dependent variables u^p . The suggested prolongation formulae (2) for $m = 1$, i.e., in the case of one dependent variable, coincide with the one given particularly for two independent variables in a recent work.⁵ In that paper,⁵ the generalized prolongation formula is proposed for the symmetry analysis of single fractional PDEs with one dependent variable and an arbitrary number of independent variables. On the other hand, in the present study, the generalized prolongation formulae are proposed for symmetry analysis of fractional systems with an arbitrary number of dependent variables and two independent variables. The proof of theorem (1) for $m = 2$, i.e., two dependent variables, has been given in our recent paper (see Ref. 8).

A. Comparison with the integer order formulae

In this subsection, the comparison of both types of prolongation formulae (1) and (2) has been done with the integer order prolongation operators. For simplification, it is performed for $m = 2$, i.e., in the case of two dependent variables say (u, v) . The symmetry operator can be written in the following form:

$$V = \xi(x, t, u, v) \partial_x + \tau(x, t, u, v) \partial_t + \eta(x, t, u, v) \partial_u + \hat{\eta}(x, t, u, v) \partial_v. \tag{4}$$

It is well known that for integer order systems of PDEs, the first order prolongation operators^{1,6} are defined as follows:

$$\begin{aligned} \eta^t &= \eta_t + (\eta_u - \tau_t) u_t + \eta_v v_t - \tau_u u_t^2 - \tau_v v_t u_t - \xi_t u_x - \xi_u u_t u_x - \xi_v v_t u_x, \\ \hat{\eta}^t &= \hat{\eta}_t + (\hat{\eta}_v - \tau_t) v_t + \hat{\eta}_u u_t - \tau_u u_t v_t - \tau_v v_t^2 - \xi_t v_x - \xi_u u_t v_x - \xi_v v_t v_x. \end{aligned} \tag{5}$$

Substituting $\alpha = 1$ and $m = 2$, expression (1) reduces to the operators as follows:

$$\begin{aligned} \eta^{(1,0)} &= \eta_t + (\eta_u - \tau_t) u_t - \tau_u u_t^2 - \tau_v v_t u_t - \xi_t u_x - \xi_u u_t u_x - \xi_v v_t u_x, \\ \hat{\eta}^{(1,0)} &= \hat{\eta}_t + (\hat{\eta}_v - \tau_t) v_t - \tau_u u_t v_t - \tau_v v_t^2 - \xi_t v_x - \xi_u u_t v_x - \xi_v v_t v_x. \end{aligned} \tag{6}$$

Obviously, the reduced formulae (6) are incomplete in comparison with (5). On the other hand, the reduced form of prolongation operators (2) for $m = 2$ and $\alpha = 1$ coincides with the integer order formulae (5). It proves the accuracy of the proposed prolongation formulae.

II. EXAMPLES

In this section, the efficiency of the proposed formulae (2) is illustrated by calculating Lie symmetries for three systems of time fractional PDEs.

A. Example 1

The first example is the same system considered in Ref. 3 and given by

$$\begin{aligned} \partial_t^\alpha u &= C^2(x) v_x, \\ \partial_t^\alpha v &= u_x. \end{aligned} \tag{7}$$

Using the proposed prolongation operators (2) in the invariance conditions given in Ref. 3, the determining equations can be obtained as follows:

$$\begin{aligned}
 \left(\frac{\alpha}{n}\right) \frac{\partial^n \hat{\eta}_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1} \tau &= 0, \\
 \left(\frac{\alpha}{n}\right) \frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1} \tau &= 0, \\
 D_t^n(\xi) &= 0, \\
 \tau_x = \tau_u = \tau_v &= 0, \\
 \eta_v - \hat{\eta}_u C^2(x) &= 0, \\
 (\eta_u - \alpha \tau_t) C^2(x) - 2C(x) \dot{C}(x) \xi - C^2(x) \hat{\eta}_v + C^2(x) D_x \xi &= 0, \\
 \hat{\eta}_v - \alpha \tau_t - \eta_u + D_x \xi &= 0, \\
 \partial_t^\alpha \eta - u \partial_t^\alpha \eta_u - v \partial_t^\alpha \eta_v - C^2(x) \hat{\eta}_x + \mu_1^1 + \mu_2^1 &= 0, \\
 \partial_t^\alpha \hat{\eta} - u \partial_t^\alpha \hat{\eta}_u - v \partial_t^\alpha \hat{\eta}_v - \eta_x + \mu_1^2 + \mu_2^2 &= 0.
 \end{aligned} \tag{8}$$

The determining equations obtained above are generalized than those in Ref. 3. Solving these determining equations, the set of solutions is as follows:

$$\xi = C(x) \left(\int \frac{C_1}{C(x)} dx + C_2 \right), \quad \tau = \frac{C_1 t}{\alpha}. \tag{9}$$

Here, for brevity, the values for η and $\hat{\eta}$ are solved only for the following two cases:

Case 1. If $C_1 = C_2 = 0$, $C(x)$ is arbitrary, integrating the determining equations gives the following symmetries:

$$\xi = 0, \quad \tau = 0, \quad \eta = C_3 u + (C_4 v) C^2(x) + f(x, t), \quad \hat{\eta} = C_3 v + C_4 u + g(x, t), \tag{10}$$

where C_3, C_4 are the arbitrary constants, and functions f, g satisfy the following:

$$\begin{aligned}
 \partial_t^\alpha f &= C^2(x) g_x, \\
 \partial_t^\alpha g &= f_x.
 \end{aligned} \tag{11}$$

Case 2. If $C_1 \neq 0, C_2 \neq 0$, and $C(x) = 1$ then the set of solutions is as follows:

$$\xi = C_1 x + C_2, \quad \tau = \frac{C_1 t}{\alpha}, \quad \eta = C_3 u + C_4 v + f(x, t), \quad \hat{\eta} = C_3 v + C_4 u + g(x, t), \tag{12}$$

where C_1, C_2, C_3, C_4 are the arbitrary constants, and the functions f, g must satisfy the following:

$$\begin{aligned}
 \partial_t^\alpha f &= g_x, \\
 \partial_t^\alpha g &= f_x.
 \end{aligned} \tag{13}$$

Clearly, the symmetries derived in both the cases are more generalized than the symmetries obtained in Ref. 3. Similarly, for the remaining cases in Ref. 3, the generalized form of symmetries can be calculated using the suggested formulae. Note that for the sake of simplicity, the linearity of η and $\hat{\eta}$ in both the dependent variables u, v has been assumed.

B. Example 2

The second considered time fractional system of PDEs is as follows:

$$\begin{aligned}
 \partial_t^\alpha u + 2v(u_x^2 + v_x^2) &= 0, \\
 \partial_t^\alpha v - 2u(u_x^2 + v_x^2) &= 0.
 \end{aligned} \tag{14}$$

The invariance criterion with the infinitesimal operator V in (4) can be written in the following form:

$$\begin{aligned} \left[\eta^{(\alpha,0)} + 2\hat{\eta}(u_x^2 + v_x^2) + 2v(2u_x\eta^x + 2v_x\hat{\eta}^x) \right] \Big|_{(14)} &= 0, \\ \left[\hat{\eta}^{(\alpha,0)} - 2\eta(u_x^2 + v_x^2) - 2u(2u_x\eta^x + 2v_x\hat{\eta}^x) \right] \Big|_{(14)} &= 0, \end{aligned} \quad (15)$$

where η^x and $\hat{\eta}^x$ are the first order prolongation operators.

Using the prolongation operators (1) in (15) and solving the resulting determining equations, the solutions can be divided into the following cases:

Case 3 $\alpha = \frac{1}{2}$. In this case, the Lie point symmetries are obtained as follows:

$$\xi = C_1x + C_2, \quad \tau = \tau(t), \quad \eta = \frac{u}{2} \left(2C_1 - \frac{\tau_t}{2} \right), \quad \hat{\eta} = \frac{v}{2} \left(2C_1 - \frac{\tau_t}{2} \right), \quad (16)$$

where C_1 and C_2 being the arbitrary constants and $\tau(0) = 0$.

Case 4 $\alpha \neq \frac{1}{2}, \tau_{tt} = 0$. In this case, the obtained symmetries are of the following form:

$$\xi = C_1x + C_2, \quad \tau = C_3t, \quad \eta = \frac{u}{2}(2C_1 - \alpha C_3), \quad \hat{\eta} = \frac{v}{2}(2C_1 - \alpha C_3), \quad (17)$$

where C_1, C_2 , and C_3 are the arbitrary constants.

On the other hand, taking advantage of the suggested operators (2) in conditions (15), the Lie point symmetries are obtained as follows:

Case 5 $\alpha = \frac{1}{2}$. In this case, the obtained Lie point symmetries are as follows:

$$\xi = C_1x + C_2, \quad \tau = \tau(t), \quad \eta = C_4v + \frac{u}{2} \left(2C_1 - \frac{\tau_t}{2} \right), \quad \hat{\eta} = -C_4u + \frac{v}{2} \left(2C_1 - \frac{\tau_t}{2} \right). \quad (18)$$

Case 6 $\alpha \neq \frac{1}{2}, \tau_{tt} = 0$. In this case, the symmetries are obtained in the following form:

$$\xi = C_1x + C_2, \quad \tau = C_3t, \quad \eta = C_4v + \frac{u}{2}(2C_1 - \alpha C_3), \quad \hat{\eta} = -C_4u + \frac{v}{2}(2C_1 - \alpha C_3), \quad (19)$$

where C_1, C_2, C_3 , and C_4 being the arbitrary constants. Clearly, symmetries (18) and (19) are more generalized than (16) and (17). It is also important to note that symmetries (18) and (19) are equivalent to the integer order symmetries of system (14) for $\alpha = 1$, proving the validity of the introduced formulae (2).

C. Example 3

The third time fractional system is given in following form:

$$\begin{aligned} \partial_t^\alpha u + v_x + 2v(u^2 + v^2) &= 0, \\ \partial_t^\alpha v - u_x - 2u(u^2 + v^2) &= 0. \end{aligned} \quad (20)$$

The invariance conditions are of the following forms:

$$\begin{aligned} \left[\eta^{(\alpha,0)} + \hat{\eta}^x + 2\hat{\eta}(u^2 + v^2) + 2v(2u\eta + 2v\hat{\eta}) \right] \Big|_{(20)} &= 0, \\ \left[\hat{\eta}^{(\alpha,0)} - \eta^x - 2\eta(u^2 + v^2) - 2u(2u\eta + 2v\hat{\eta}) \right] \Big|_{(20)} &= 0. \end{aligned} \quad (21)$$

Using the prolongation operators (1), the Lie point symmetries can be obtained as follows:

$$\xi = C_1x + C_2, \quad \tau = \frac{C_1t}{\alpha}, \quad \eta = -\frac{C_1u}{2}, \quad \hat{\eta} = -\frac{C_1v}{2}, \quad (22)$$

where C_1 and C_2 being the arbitrary constants. Applying the prolongation formulae (2) proposed in this study, the investigated symmetries are of the following form:

$$\xi = C_1x + C_2, \quad \tau = \frac{C_1t}{\alpha}, \quad \eta = -\frac{C_1u}{2} + C_3v, \quad \hat{\eta} = -\frac{C_1v}{2} - C_3u, \quad (23)$$

where C_1 , C_2 , and C_3 are the arbitrary constants. Symmetries (23) are general than (22) and also equivalent with the integer order symmetries of system (20) for $\alpha = 1$. Here, for simplicity, the linearity of η and $\hat{\eta}$ in both the dependent variables has been chosen.

III. CONCLUSION

In the present study, the general fractional order prolongation formulae (2) for time fractional systems of $(1 + 1)$ -dimensional PDEs having an arbitrary number of dependent variables have been proposed. By the derivation of Lie symmetries of three systems of FPDEs, it is proved that general form of symmetries can be obtained using operators (2) rather than formulae (1). Hence, it is proved that operators (2) are more efficient and accurate for executing Lie symmetry analysis of time fractional systems of PDEs.

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